Linear Map 1

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ABSTRACT: References are of course my favorite [1, 2] and additionally, [3–5]. For the basis, I think [6] explains very well.

Contents

1 Linear map: Homomorphism of vector spaces

We have learnt the most important map, homomorphisms, which is the structure-preserving map in the set and group theory. Now we want to know this important map between the vector space. As we have seen, the vector space is defined as a set whose addition and s-multiplication (with some field) is well-defined. Therefore, the homomorphism between the vector space needs to preserve the addition and s-multiplication. We have special name for the vector space homomorphism. Since its another name is linear space, it is called the linear map (or linear operator)

Box 1.1: Linear Map (or Linear Operator or linear transformation)

Let $(V, +_V, \heartsuit_V)$ and $(W, +_W, \heartsuit_W)$ be $\mathbb{K}$-vector spaces. Then, a map $\phi : V \to W$ is called a *linear map* (or *linear operator* or *linear transformationlinear transformation*) if it is

- *additive:* $\forall\, v_1, v_2 \in V : \phi(v_1 +_V v_2) = \phi(v_1) +_W \phi(v_2)$;
- *homogeneous:* $\forall\, a \in \mathbb{K} : \forall\, v \in V : \phi(a \heartsuit_V v) = a \heartsuit_W \phi(v)$.

Like group theory, you can also use diagrammatic expression

$$G \xrightarrow{\phi} H \ . \tag{1.1}$$

Note that the addition and s-multiplication change their domains.

Theorem 1.1. Let $(V, +_V, \heartsuit_V)$ and $(W, +_W, \heartsuit_W)$ be $\mathbb{K}$-vector spaces.The map $\phi : V \to W$ is a linear map if

$$\forall\, a, b \in \mathbb{K} : \forall\, v_1, v_2 \in V : \phi(a \heartsuit_V v_1 +_V b \heartsuit_V v_2) = a \heartsuit_W \phi(v_1) +_W b \heartsuit_W \phi(v_2). \tag{1.2}$$

Theorem 1.2. Let $0_V \in V$ and $0_W \in W$ be the zero vectors for each vector space. For the linear map $\phi : V \to W$, $\phi(0_V) = 0_W$.

Example 1.2.1 (zero map)**.** The zero map $\mathsf{o} : V \to W$ by $v \mapsto 0_W$ is a linear map.

Example 1.2.2 (identity map)**.** The identity map $\mathrm{id}_V : V \to V$ by $v \mapsto v$ is a linear map.

The following two example domain can be extended to general differentiable functions, but we will keep it for later.

Example 1.2.3 (Differentiation)**.** Let $P(n)$ be the vector space of nth order polynomial. Then the differentiation $D : P(n) \to P(n-1)$ is a linear map.

Example 1.2.4 (Integration)**.** Let $P(n)$ be the vector space of nth order polynomial. Then the integration $I : P(n) \to P(n+1)$ by

$$\sum_{i=0}^{n} a_i x^i \mapsto \sum_{n=0}^{n} \frac{a_i}{(i+1)!} x^{i+1} \tag{1.3}$$

is a linear map.

We now want to say what is the meaning of two spaces are the same in the sense of linear algebra.

Box 1.2: (Vector space) Isomorphic and Isomorphism

Two vector spaces V and W are called *(Vector space) isomorphic* if there exists a bijective linear map $\phi\colon V \to W$. In this case, we write $V \cong_{\text{vec}} W$. The map ϕ is called an *vector space isomorphism* or *linear isomorphism*.

Theorem 1.3. Let V, W be vector spaces. $V \cong_{\text{vec}} W$ iff there is a linear map $\phi : V \to W$ which has an inverse linear map $\phi^{-1} : W \to V$.

1.1 Linear Map is a Map

In here, we will focus on the property of linear map which arises because it is a map. The standard terminology for the map holds such as domain, target, image, and pre-image. Just as a review, I rewrite the definition of image and preimage explicitly with linear map.

Definition 1.1 (image)**.** Let $\phi : V \to W$ be a linear map. For the set $U \subseteq V$, the set $\phi(U) \equiv \mathrm{im}_\phi(U) := \{\phi(v) \in W \mid v \in U\}$ is called the *image* of U under ϕ. When $U = V$, we will simply write $\operatorname{im}\phi \coloneqq \mathrm{im}_\phi(V)$ and call just the image of ϕ.

Definition 1.2 (pre-image)**.** Let $\phi\colon V \to W$ be a linear map and let $U \subseteq W$. Then we define the set:

$$\mathrm{preim}_\phi(U) := \{v \in V \mid \phi(v) \in U\} \tag{1.4}$$

called the *pre-image* of U under ϕ.

In the theorem above, we see that every zero vector maps to a zero vector. Therefore, zero vector on the target space is always in the image. The preimage of zero vector has a special name.

Definition 1.3 (Kernel)**.** The *kernel*(or *null space*) of a linear map $\phi : V \to W$ is the preimage of zero vector. That is

$$\ker\phi \coloneqq \mathrm{preim}_\phi(\{0_W\}) := \{u \in V \mid \phi(v) = 0_W\} \tag{1.5}$$

Example 1.3.1. For zero map $\mathsf{o} : V \to W$, kernel is domain, $\ker\phi = V$.

Theorem 1.4. If a linear map $\phi : V \to W$ is injective, $\ker\phi = \{0_V\}$.

The dimension of kernel and image has very important relation in finite dimension.

Box 1.3: Fundamental Theorem of Linear Maps (Rank-Nullity Theorem)

Let V, W be a finite-dimensional vector space. For a linear map $\phi : V \to W$, the *fundamental theorem of linear maps* is the the dimension of the domain vector space is the same with the sum of the dimension of kernel and the dimension of image,

$$\dim V = \dim(\ker \phi) + \dim(\operatorname{im} \phi) \tag{1.6}$$

The dimension of kernel $\dim(\ker \phi)$ is called the *nullity* of ϕ and the dimension of image of ϕ, $\dim(\operatorname{im} \phi)$ is called the *rank* of ϕ. Therefore, the theorem has another name "rank-nullity theorem".

Proof. Let $\phi : V \to W$ be a linear map. Then we can define the equivalence relation $\sim$ such that

$$\forall\, v, v' \in V : (v' \sim v \quad :\Leftrightarrow \quad \exists\, n \in \ker V : v' = v + n). \tag{1.7}$$

With this, we can define the quotient space $V/(\ker \phi) \coloneqq V/\sim$ and natural isomorphism $\bar{\phi} : V/(\ker \phi) \to \operatorname{im} \phi$. The existence of this is guaranteed by the *first isomorphism theorem* (or *fundamental theorem of homomorphism*)[1] Since, there is an isomorphism,

$$\operatorname{im} \phi \cong_{\text{vec}} V/(\ker \phi), \tag{1.8}$$

which implies

$$\dim(\operatorname{im} \phi) = \dim \big(V/(\ker \phi)\big). \tag{1.9}$$

Note also that from the definition of equivalence relation, one gets

$$V = (V/\sim) \oplus [0] = \big(V/(\ker \phi)\big) \oplus \ker \phi, \tag{1.10}$$

which results in

$$\dim \big(V/(\ker \phi)\big) = \dim V - \dim(\ker \phi). \tag{1.11}$$

Combining above two equations results

$$\dim V = \dim(\ker \phi) + \dim(\operatorname{im} \phi). \tag{1.12}$$

□

It gives following restriction criteria of domain:

Lemma 1.5. For the linear map $\phi : V \to W$ between finite dimensional vector space V and W:

- If $\dim V > \dim W$, ϕ is not injective;
- If $\dim V < \dim W$, ϕ is not surjective.

[1] See 'Fundamental theorem on homomorphisms', 'isomorphism theorems' in wikipedia for more detail.

Proof. Let us prove the first lemma. For the injective function, $\ker\phi = \{0\}$. Therefore $\dim(\ker\phi) = 0$. By the way,

$$\begin{aligned} \dim(\ker\phi) &= \dim V - \dim(\operatorname{im}\phi) && (1.13) \\ &\geq \dim V - \dim W && (1.14) \\ &> 0 && (1.15) \end{aligned}$$

Therefore, ϕ cannot be a injective.

Now, let us prove the second lemma. For the surjective map, $\operatorname{im}\phi = W$. Therefore, $\dim(\operatorname{im}\phi) = \dim W$. By the way,

$$\begin{aligned} \dim(\operatorname{im}\phi) &= \dim V - \dim(\ker\phi) && (1.16) \\ &\leq \dim V && (1.17) \\ &< \dim W. && (1.18) \end{aligned}$$

Hence, it is not surjective. □

Moreover, we have the following powerful lemma.

Box 1.4: Dimension of Isomorphism

Let the V, W be finite-dimensional $\mathbb{K}$-vector spaces. Then,

$$\dim V = \dim W \iff V \cong_{\text{vec}} W. \tag{1.19}$$

Proof. ($\Longleftarrow$) can be shown simply by using above lemma. Since isomorphism is bijection,

- It is surjective $\implies$ $\dim V \geq \dim W$;
- It is injective $\implies$ $\dim V \leq \dim W$.

Therefore,

$$(\dim V \geq \dim W) \wedge (\dim V \leq \dim W) \iff \dim V = \dim W. \tag{1.20}$$

In other words, if there is an isomorphism $\phi : V \to W$, $\dim V = \dim W$.

($\implies$) Let us construct an injective linear map $\phi : V \to W$. We have shown that $\ker\phi = \{0_V\}$ for an injective map which means that $\dim V = 0$. From the fundamental theorem of linear maps, we can conclude that

$$\dim(\operatorname{im}\phi) = \dim V = \dim W. \tag{1.21}$$

Therefore $\operatorname{im}\phi \cong_{\text{set}} W$ which means that it is surjective map also. We can conclude that ϕ is bijection, so isomorphism. In other words, if $\dim V = \dim W$, we can construct an isomorphism between V and W. □

Remark. In the proof of ($\Longleftarrow$) we can construct an surjective map first. In fact, if $\dim V = \dim W$, following are equivalent:

- A linear map $\phi : V \to W$ is injective;

- A linear map $\phi : V \to W$ is surjective;
- A linear map $\phi : V \to W$ is bijective.

Then, how to construct that map? There is a theorem using basis.

Theorem 1.6 (Construction of Map)**.** Let $\{e_1, \ldots, e_n\}$ be a basis of V and $w_1, \ldots, w_n \in W$ are just an arbitrary vectors. Then,

$$\exists!\, \phi : V \to W \qquad \text{such that} \qquad \forall k \in \{1, \ldots, n\} : \quad \phi(e_k) = w_k \tag{1.22}$$

Note that not only exists, it is unique! Therefore, there is always a unique map which sends the basis of V to the chosen basis of W.

Proof. The first thing to show is the existence of a map. Let us define ϕ be a map which satisfies

$$\phi(c_1 e_1 + \ldots + c_n e_n) = c_1 w_1 + \ldots c_n w_n. \tag{1.23}$$

Then choosing $c_j = \delta_{jk}$, we can recover $\phi(e_k) = w_k$. Next step is to show it is a linear map. Since $\{e_1, \ldots, e_n\}$ is a basis of V,

$$\forall v \in V : \exists! v_1, \ldots, v_n \in \mathbb{K} : \quad v = \sum_n v_n e_n. \tag{1.24}$$

Then,

$$\begin{aligned} \forall\, a \in \mathbb{K} : \forall u, v \in V : \phi(au + v) &= \phi\big(a \sum_{i=1}^{n} u_i e_i + \sum_{i=1}^{n} v_i e_i\big) && (1.25)\\ &= \phi\big((au_1 + v_1)e_1 + \ldots + (au_n + v_n)e_n\big) && (1.26)\\ &= (au_1 + v_1)w_1 + \ldots + (au_n + v_n)w_n && (1.27)\\ &= a(u_1 w_1 + \ldots + u_n w_n) + (v_1 w_1 + \ldots + v_n w_n) && (1.28)\\ &= a\phi(u) + \phi(v). && (1.29) \end{aligned}$$

Therefore, ϕ is a linear map. The last step is to show that it is unique. Since it is a linear map,

$$\phi(v) = \phi(v_1 e_1 + \ldots + v_n e_n) = v_1 \phi(e_1) + \ldots + v_n \phi(e_n) = v_1 w_1 + \ldots + v_n w_n. \tag{1.30}$$

therefore, the map we choose is in fact the only map satisfying the condition. □

1.2 Linear Map Forms a Vector Space

Recall that we always want to understand concepts from the set-theory based language. Our new concept, linear map should be an element of some set.

Box 1.5: Set of Linear Maps

The set of linear map $\phi : V \to W$ is denoted $\mathcal{L}(V, W)$ or $\mathrm{Hom}(V, W)$. In symbol:[a]

$$\mathcal{L}(V, W) = \mathrm{Hom}(V, W) \coloneqq \{\phi \in \mathcal{P}(V \times W) \mid \phi : V \to W \text{ is linear } \}. \tag{1.31}$$

The later comes from the fact that the linear map is an homomorphism. Note that ϕ is a binary relation, therefore, $\phi \subseteq V \times W$.

[a]Note that the map is a relation and relation is the subset of cartesian product.

Definition 1.4 (Endomorphism)**.** Let V be a vector space. A linear map into itself is called an *endomorphism*. The set of endomorphisms are denoted $\mathrm{End}(V) \coloneqq \mathrm{Hom}(V, V)$.

Definition 1.5 (Automorphism)**.** Let V be a vector space. A linear isomorphism into itself is called an *automorphism*. The set of automorphisms are denoted $\mathrm{Aut}(V)$.

Recall that the element of a set is the object what we want to describe. When we start the linear algebra, I told that everything we treat in the linear algebra is a vector. Now you know this is a bad sentence. The underlying idea on this is that the set of objects we will meet in the linear algebra with proper addition and s-multiplication rules will form a vector space.

Box 1.6: Vector Space of Linear Maps

Let $(V, +_V, \heartsuit_V)$ and $(W, +_W, \heartsuit_W)$ be $\mathbb{K}$-vector spaces.And let us define the binary operators $\boxplus : \mathrm{Hom}(V, W) \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ and $\boxdot : \mathbb{K} \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ to satisfy

$$\forall \phi, \psi \in \mathrm{Hom}(V, W) : \forall\, v \in V : (\phi \boxplus \psi)(v) = \phi(v) +_W \psi(v); \tag{1.32}$$

and

$$\forall a \in \mathbb{K} : \forall \phi \in \mathrm{Hom}(V, W) : \forall v \in V : (a \boxdot \phi)(v) = a \heartsuit_W \phi(v), \tag{1.33}$$

Then, the triple $(\mathrm{Hom}(V, W), \boxplus, \boxdot)$ is a *vector space.*

In the previous section, we learnt that the dimension is useful in vector space. Then what is the dimension of $\mathrm{Hom}(V, W)$? Before going on, let us see some definition for completeness and clarify the confusion. The map is just the subset of cartesian product of domain and target. But, there is also the Cartesian products of vector spaces.

Definition 1.6 (Cartesian Product)**.** Let $(V_1, +_{V_1}, \heartsuit_{V_1}), \ldots, (V_m, +_{V_m}, \heartsuit_{V_m})$ be a $\mathbb{K}$-vector space. The Cartesian product of them is

$$V_1 \times \ldots \times V_m \coloneqq \{(v_1, \ldots, v_m) \mid v_1 \in V_1 \wedge \ldots \wedge v_m \in V_m\}. \tag{1.34}$$

In this case, we have unique addition and multiplication,

$$+ : \quad (V_1 \times \ldots \times V_m) \times (V_1 \times \ldots \times V_m) \to V_1 \times \ldots \times V_m; \tag{1.35}$$

$$\heartsuit : \quad \mathbb{K} \times (V_1 \times \ldots \times V_m) \to V_1 \times \ldots \times V_m, \tag{1.36}$$

which defined to satisfy: $\forall a \in \mathbb{K} : \forall (u_1, \ldots, u_n), (v_1, \ldots, v_n) \in V_1 \times \ldots \times V_m$:

$$(u_1, \ldots, u_n) + (v_1, \ldots, v_n) \coloneqq (u_1 +_{V_1} v_1, \ldots, u_m +_{V_m} v_m); \tag{1.37}$$
$$a\heartsuit(v_1, \ldots, v_n) \coloneqq (a\heartsuit_{V_1} v_1, \ldots, a\heartsuit_{V_m} v_m). \tag{1.38}$$

We call the triple $(V_1 \times \ldots \times V_m, +, \heartsuit)$ is a *Cartesian product* of vector spaces $V_1 \times \ldots \times V_m$. Like always, we lazily just call $V_1 \times \ldots \times V_m$ is a cartesian products.

Remark. Note that for defining the Cartesian product, we do not need to say about the larger vector space, because we already have a unique addition and multiplication. In other words, each vector spaces $V_1, \ldots, V_m$ does not need to be a subspace of some larger space. On the other hand, when we say the direct sum of vector space, we always need a larger space to have a well-defined addition and multiplication.

Remark. But, since $\forall i \in \{1, \ldots, m\} : V_i \subseteq V_1 \times \ldots \times V_m$, we might can relate the direct sum and the Cartesian product. In fact, for the *finite* number of vector spaces the addition and s-multiplication in both definitions are the same.[2] Therefore,

$$V_1 \times \ldots \times V_m = V_1 \oplus \ldots \oplus V_m. \tag{1.39}$$

Theorem 1.7 (Dimension of Cartesian Products of vector spaces)**.** The dimension of Cartesian products of vector spaces are the sum of dimensions of them:

$$\dim(V_1 \times \ldots \times V_m) = \dim V_1 + \ldots + \dim V_m. \tag{1.40}$$

Remark. Even though we use just name Cartesian product for both set and vector space, there is clear distinction. Note that in the definition 1.5, the Cartesian product is just the Cartesian product of sets. The addition and s-multiplication is not the same.

The power set is very big set. So we might want to check the dimension of the set of linear maps. We can guess that

$$\dim V + \dim W \leq \dim\big(\mathrm{Hom}(V, W)\big) \leq \dim \mathcal{P}(V \times W). \tag{1.41}$$

In fact, we have the following.

Box 1.7: Dimension of set of Linear Map

The dimension of the set of linear maps are just the multiplication of the dimension of each vector spaces:

$$\dim\big(\mathrm{Hom}(V, W)\big) = (\dim V)(\dim W). \tag{1.42}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis for V and $\{f_1, \ldots, f_m\}$ be a basis for W. Then, we have a unique map for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$:

$$\phi(e_i) = f_j. \tag{1.43}$$

Since there is nm independent choices, we have nm independent vectors in $\mathrm{Hom}(V, W)$. Therefore, $\dim(\mathrm{Hom}\,(V, W)) = nm = (\dim V)(\dim W)$ □

[2] They are not the same in infinite dimensions.

There are many important properties in the linear map. Because is has a property of map and also has a property of vector space. I prefer to tell all the stories of linear map first. But because many books, including [7], talks about linear maps after treating our younger time concepts which are non-necessary but familiar, we will also keep more details on the linear maps later.[3]

2 Dual Space

In the vector space example, we have seen that $\mathbb{R}$ is a $\mathbb{R}$-vector space, and $\mathbb{C}$ is a $\mathbb{C}$ vector space. In other words, scalars form a vector space on it. In this sense, we can think about the special linear map which sends vectors to the scalar.

Box 2.1: Linear Form (or Linear Functional or One-Form or Covector)

Let V be a $\mathbb{K}$-vector space. A linear map $\phi : V \to \mathbb{K}$ is called a *linear form* (or *linear functional*, or *one-form*, or *covector*. That is

$$\phi \in \mathrm{Hom}(V, \mathbb{K}). \tag{2.1}$$

Example 2.0.1 (zero map)**.** The zero map $\mathfrak{o} : V \to \mathbb{K}$ by $v \mapsto 0$ is a one-form.

Example 2.0.2 (Integration)**.** Let F be the vector space of real functions from $\mathbb{R} \to \mathbb{R}$. Then the definite integration $I : F \to \mathbb{R}$ by

$$f(x) \mapsto \int_0^1 f(x)\mathrm{d}x \tag{2.2}$$

is a one-form.

Box 2.2: Dual Space

Let V be a $\mathbb{K}$-vector space. The set of all one-forms of V is called a *dual space* and denoted V^*. In other words,

$$V^\vee := V^* := \mathrm{Hom}(V, \mathbb{K}). \tag{2.3}$$

There is a special one-form which has also the name coordinate form or dual form.

Box 2.3: Dual Form (or Coordinate Form)

For given basis $\{e_1, \ldots, e_n\}$ of V, the one-forms $\alpha^i : V \to \mathbb{K}$, $i \in \{1, \ldots, n\}$ which satisfies

$$\alpha^i(e_j) = \delta^i_j, \tag{2.4}$$

where δ^i_j is the kronecker delta symbol, are called a *coordinate forms* or *dual form* for the given basis.

[3] I am little bit unsatisfactory about this flow because it seems less logical for me. But it might be more pedagogical.

Remark. Recall that if $\{e_1, \ldots, e_n\}$ is the basis of V,

$$\forall\, v \in V : \forall\, i \in \{1, \ldots, n\} : \exists!v^i \in \mathbb{K} : \quad v = \sum_{i=1}^{n} v^i e_i. \tag{2.5}$$

The dual form is called the coordinate form because

$$\alpha^i(v) = \alpha^i(v^1 e_1 + \ldots + v^n e_n) = v^1 \alpha^i(e_1) + \ldots + v^n \alpha^i(e_n) = \sum_{j=1}^{n} v^j \delta^i_j = v^j. \tag{2.6}$$

If you chose e_j to be the orthonormal basis in Euclidean space, it is more clear.

Remark. Note, however, that we do not define the inner product. In other words, we do not have definition of angle (orthogonality is not well-defined) or length (normalisation is not well-defined).

Recall that the set of linear maps with addition and s-multiplication is a vector space. Since the one-form is the special case of linear map, the set of one-forms with addition and s-multiplication above also is a vector space. In fact, dual forms is a basis for this vector space, and that's why they have another name *dual basis.*

Box 2.4: Dual basis

For given basis $\{e_1, \ldots, e_n\}$ of V, the set of dual forms $\{\alpha^i : V \to \mathbb{K}\}$ such that

$$\alpha^i(e_j) = \delta^i_j \qquad i, j \in \{1, \ldots, n\} \tag{2.7}$$

is a basis for the dual space $V^\vee$

Proof. Let us first show that $\{\alpha^1, \ldots, \alpha^n\}$ spans $V^\vee$. Let the basis expension for $v \in V$ to be

$$v = \sum_{i=1}^{n} v^i e_i. \tag{2.8}$$

Then

$$\forall\, f \in V^\vee : f(v) = f\Big(\sum_{i=1}^{n} v^i e_i\Big) = \sum_{i=1}^{n} v^i f(e_i). \tag{2.9}$$

Recall that $v^i = \alpha^i(v)$. Therefore,

$$\forall\, f \in V^\vee : f(v) = \Big(\sum_{i=1}^{n} f(e_i)\alpha^i\Big)(v). \tag{2.10}$$

where the summation symbol is now for the addition on $V^\vee$. Since $f \in V^\vee$, $f(e_i)$ is just a scalar. Therefore, we have linear combination expression:

$$\forall\, f \in V^\vee : f = \sum_{i=1}^{n} f(e_i)\alpha^i. \tag{2.11}$$

In other words,

$$V^\vee = \mathrm{span}(\alpha^1, \dots, \alpha^n). \tag{2.12}$$

The next step is to show that α^i are linearly independent. It is to check

$$\sum_{i=1}^{n} c_i \alpha^i = 0 \qquad \overset{?}{\Longleftrightarrow} \qquad c_1 = \ldots = c_n = 0. \tag{2.13}$$

Let us apply the left equation on e_j.

$$\sum_{i=1}^{n} c_i \alpha^i(e_j) = \sum_{i=1}^{n} c_i \delta^i_j = c_j = 0. \tag{2.14}$$

Since, this should be true for all $j \in \{1, \dots, n\}$, we can conclude that

$$\sum_{i=1}^{n} c_i \alpha^i = 0 \qquad \Longrightarrow \qquad c_1 = \ldots = c_n = 0. \tag{2.15}$$

On the other hand,

$$\sum_{i=1}^{n} c_i \alpha^i = 0 \qquad \Longleftarrow \qquad c_1 = \ldots = c_n = 0 \tag{2.16}$$

is trivial. Therefore, we can conclude that they are linearly independent. Since we show that $\{\alpha^1, \dots, \alpha^n\}$ is linearly independent spanning set of $V^\vee$, it is a basis for $V^\vee$. □

Definition 2.1 (annihilator)**.** Let $U \subseteq V$ be a subspace of V. The *annihilator* of U in V, denoted $U^\perp$, is the set of one-forms on V which vanishes on U i.e.,

$$U^\perp \coloneqq \{\phi \in V^\vee \mid \forall\, u \in U : \phi(u) = 0\}. \tag{2.17}$$

Remark. It is similar to the $\ker\phi$. But note that $\ker\phi \subseteq V$. On the other hand, $U^\perp \subseteq V^\vee$.

Definition 2.2 (double dual)**.** Since, $V^\vee$ is a vector space, we can also think about the dual space of it which is called *double dual*, *bidual* or *dual of dual*. The notation is

$$V^{\vee\vee} \coloneqq (V^\vee)^\vee \coloneqq V^{**} \coloneqq (V^*)^* \coloneqq \mathrm{Hom}(V^\vee, \mathbb{K}). \tag{2.18}$$

Theorem 2.1 (Double dual isomorphism)**.** The double dual of vector space is isomorphic to the original vector space.

$$V^{\vee\vee} \cong_{\text{vec}} V. \tag{2.19}$$

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