Linear Algebra and Vector Space

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Abstract: References are of course my favorite [1, 2] and additionally, [3–5]. For the basis, I think [6] explains very well.

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1 Vector Space (Linear Space)

Let us recall our baby time. The most basic algebra we learnt is addition, then we learnt multiplication as an iterative addition. Vector space is nothing but the addition and s-multiplication[1] is well-defined.

Box 1.1: Vector Space (linear space)

Let the triple $(\mathbb{K}, +_{\mathbb{K}}, \cdot_{\mathbb{K}})$ be a field. A *vector space* (or *linear space*) V on a field $\mathbb{K}$ is the triple $(V, +, \heartsuit)$ where V is a set, $+ : V \times V \to V$ and $\heartsuit : \mathbb{K} \times V \to V$ are the maps which satisfies the following axioms:[a]
$(V.+)$is an abelian group:

i) $\forall\, v_1, v_2 \in V : v_1 + v_2 = v_2 + v_1$;
ii) $\forall\, v_1, v_2, v_3 \in V : (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$;
iii) $\exists 0 \in V : \forall v \in V : v + 0 = 0 + v = v$;
iv) $\forall v \in V : \exists - v \in V : v + (-v) = (-v) + v = 0$;

$\heartsuit$ is called an *s-multiplication*

v) $\forall a, b \in \mathbb{K} : \forall v \in V : (a \cdot_{\mathbb{K}} b) \heartsuit v = a \heartsuit (b \heartsuit c)$;
vi) $\forall\, a, b \in \mathbb{K} : \forall\, v \in V : (a +_{\mathbb{K}} b) \heartsuit v = a \heartsuit v + b \heartsuit v$;
vii) $\forall\, a \in \mathbb{K} : \forall\, v_1, v_2 \in V : a \heartsuit (v_1 + v_2) = a \heartsuit v_1 + a \heartsuit v_2$;
viii) For $1 \in \mathbb{K} : \forall\, v \in V : 1 \heartsuit v = b \heartsuit a$.

We sometimes simply call $\mathbb{K}$-*vector space* (or $\mathbb{K}$-*linear space*) V or even just *vector space* (or *linear space*) V.

[a] We usually do not write $\heartsuit$ and omit it. But, in this section, we write it explicitly to emphasise what is the domain.

Remark. For the addition of inverse element of addition, we usually omit the symbol $+$. That is for $v, w \in V$,

$$v - w :\Leftrightarrow v + (-w) \tag{1.1}$$

Remark. Particularly, $\mathbb{R}$-vector space is called a *real vector space* and $\mathbb{C}$-vector space is called a *complex vector space*.

Example 1.0.1 (Arrows)**.** Let the set of all arrows in our space be V. The addition and multiplication rules what we learnt in physics 1 class forms a real vector space V.

Example 1.0.2 (Numbers)**.** $\mathbb{R}$ and $\mathbb{C}$ are both real vector space with the addition and multiplication to be the same with the operation we know. $\mathbb{C}$ with the addition and multiplication to be the same with the operation we know is also complex vector space. But $\mathbb{R}$ cannot be a complex vector space.

[1] Do not confuse with product.

Definition 1.1 (Vector)**.** A *vector* is just an element of V where V is vector space.

Remark. You might heard to answer the question something is a vector or not. Note that the same set (or element) can be a vector space or not by choosing the field or operation. Therefore, in fact, asking something is a vector or not is mathematically inadequate question.

Example 1.0.3 (n-tuples)**.** A set of n-tuples of real numbers $\mathbb{R}^n$ with component-wise addition and multiplication defined by multiplying the number to each components also forms a real vector space. Similarly, complex numbers $\mathbb{C}^n$ with the same operations forms a complex vector space.

Example 1.0.4 (Matrices)**.** A set of all matrices $\mathrm{Mat}_{m\times n}$ with usual matrix addition and multiplication with numbers forms a vector space.

Example 1.0.5. A set of n-th order polynomials with real coefficients and the addition and multiplication of them in usual sense forms a real vector space.

Example 1.0.6 (Functions)**.** The set of all real functions with point-wise addition and multiplication forms a real vector space. Similarly, the set of all complex functions with point-wise addition and multiplication forms a complex vector space.

Since we have learnt the set, you might want to check what is the *intersection* or *union* of vector spaces. It is not, however, possible to say the resulting space is what because we do not know the operation in that union or intersection of that vector spaces.

1.1 Subspace

On the other hand, we can define the addition and multiplication for the subset of vector space unambiguously from the original operation.

> **Box 1.2: Subspace**
>
> Let $(V, +_V, \heartsuit_V)$ be a $\mathbb{K}$-vector space. For a set $U \subset V$, $(U, +_V, \heartsuit_V)$ is called a *subspace* of V if it is vector space. We will also write just U is a subspace of V[a].
>
> [a] $_V$ is used to emphasise the operation is defined in the original set V

Note that the operation is defined in the original vector space V. Since operations are already commutative, associative and distributive, we need to check if the operations are closed in U and zero vector 0 is in U $(0 \in U)$.

Theorem 1.1. *U is a subspace of $\mathbb{K}$-vector space $(V, +, \heartsuit)$ iff*

$$\forall\, a, b \in \mathbb{K} : \forall\, u_1, u_2 \in U : a \heartsuit u_1 \, + \, b \heartsuit u_2 \in U. \tag{1.2}$$

I omit the trivial operations in the following examples. I strongly recommend readers to check what is the addition and multiplication in each example.

Example 1.1.1. A complex vector space $\mathbb{R}$ defined above example is a subset of a complex vector space $\mathbb{C}$.

Example 1.1.2. A real number $\mathbb{R}$ is a subspace of $\mathbb{R}^2$.

Example 1.1.3. The set of continuous(differentiable, analytic) real functions are the subspace of the set of all real functions.

Example 1.1.4. A finite line segment $[a, b] \subset \mathbb{R}$ is not a subset of plane $\mathbb{R}^2$.

Using the subspace, we can define the intersection or union of the subspaces unambiguously.

Definition 1.2 (Intersection of subspaces)**.** Let $(U_1, +, \heartsuit)$ and $(U_2, +, \heartsuit)$ are the subspace of V. Then the intersection $U_1 \cap U_2$ is called the *intersection* of subspaces U_1 and U_2.

Theorem 1.2. Let $(U_1, +, \heartsuit)$ and $(U_2, +, \heartsuit)$ are the subspace of V. Then $(U_1 \cap U_2, +, \heartsuit)$ is also a subspace of V.

Proof. We need to check closure.

$$\forall\, a, b \in \mathbb{K} : \forall\, u_1, u_2 \in U_1 : a \heartsuit u_1 + b \heartsuit u_2 \in U_1 \tag{1.3}$$

$$\forall\, a, b \in \mathbb{K} : \forall\, u_1, u_2 \in U_2 : a \heartsuit u_1 + b \heartsuit u_2 \in U_2 \tag{1.4}$$

Therefore,

$$\forall\, a, b \in \mathbb{K} : \forall\, u_1, u_2 \in U_1 \cap U_2 : \big(a \heartsuit u_1 + b \heartsuit u_2 \in U_1\big) \wedge \big(a \heartsuit u_1 + b \heartsuit u_2 \in U_2\big) \tag{1.5}$$

We can rewrite this

$$\forall\, a, b \in \mathbb{K} : \forall\, u_1, u_2 \in U_1 \cap U_2 : a \heartsuit u_1 + b \heartsuit u_2 \in U_1 \cap U_2 \tag{1.6}$$

□

Definition 1.3 (Union of subspaces)**.** Let $(U_1, +, \heartsuit)$ and $(U_2, +, \heartsuit)$ are the subspace of V. Then the union $U_1 \cup U_2$ is called the *union* of subspaces U_1 and U_2.

Theorem 1.3. Let $(U_1, +, \heartsuit)$ and $(U_2, +, \heartsuit)$ are the subspace of V. In general, $(U_1 \cup U_2, +, \heartsuit)$ is not a subspace of V.

Proof. Let $u_1 \in U_1$ and $u_2 \in U_2$ are both nonzero vectors. Then $u_1 + u_2 \notin U_1 \wedge u_1 + u_2 \notin U_2$. Therefore, $u_1 + u_2 \notin U_1 \cup U_2$ and we can conclude that the addition is not closed. □

It is shameful that not all the unions of subspaces are subspaces. In the proof, we notice that the problem occurs because the addition is not closed. We can extend the set to make the addition to be closed.

Definition 1.4 (Sum of Subspaces)**.** Let $(U_1, +_V, \heartsuit_V)$ and $(U_2, +_V, \heartsuit_V)$ are the subspace of V. Then, the *sum* of U_1 and U_2 is

$$U_1 + U_2 \coloneqq \{u_1 +_V u_2 \in V \mid u_1 \in U_1 \wedge u_2 \in U_2\}. \tag{1.7}$$

The triple $(U_1 + U_2, +_V, \heartsuit_V)$ is now a subspace of V. We also simply say that $U_1 + U_2$ is a subspace of V. One can extend the sum of many subspaces by iteratively sum each spaces.

Theorem 1.4. Let $(U_1, +_V, \heartsuit_V)$ and $(U_2, +_V, \heartsuit_V)$ are the subspace of V. Then the intersection $(U_1 \cup U_2, +, \heartsuit)$ is a subspace of sum $(U_1 + U_2, +_V, \heartsuit_V)$.

Remark. Since both U_1 and U_2 contains 0, they cannot be disjoint. But what if there is other nonzero $v \in U_1 \cap U_2$? In that case, the expression in the above definition is redundant because

$$u_1 + u_2 = (u_1 + v) + (u_2 - v) \tag{1.8}$$

and

$$u_1 \in U_1 \wedge u_2 \in U_2 \iff (u_1 + v) \in U_1 \wedge (u_2 - v) \in U_2. \tag{1.9}$$

Since we do not like the redundant expression, we give the special name for the sum of subspaces when the expression is unique

Box 1.3: Direct Sum

The sum of subspaces of vector space is called *direct sum* of subspaces if their intersection only contains 0. To emphasise it, we use $\oplus$ instead of $+$:

$$U_1 \oplus U_2 \coloneqq \big\{u_1 +_V u_2 \in V \mid u_1 \in U_1 \wedge u_2 \in U_2 \wedge (U_1 \cap U_2 = \{0\})\big\}. \tag{1.10}$$

where U_1 and U_2 are subspace of V.

1.2 Factor space (Quotient space)

Using equivalence relation $\sim$, we could partition the set. Note that one can also define the addition and s-multiplication uniquely from the original vector space. Therefore, it is possible to partition the vector space using the equivalence relation.

Box 1.4: Quotient Space (Factor Space)

Let $(V, +, \heartsuit)$ be a $\mathbb{K}$-vector space and $\sim$ be an equivalence relation. Let us define the addition $+_\sim : (V/\sim) \times (V/\sim) \to (V/\sim)$ and s-multiplication $\heartsuit_\sim : \mathbb{K} \times (V/\sim) \to (V/\sim)$ to satisfy

$$\forall a, b \in \mathbb{K} : \forall v_1, v_2 \in V : [a \heartsuit v_1 + b \heartsuit v_2] = a \heartsuit_\sim [v_1] +_\sim b \heartsuit_\sim [v_2]. \tag{1.11}$$

Then the vector space $(V/\sim, +_\sim, \heartsuit_\sim)$ is called a *quotient space* of V. We just write $V/\sim$ is a quotient space of V

Example 1.4.1. For $\mathbb{R}^2$, let us define the equivalence relation to be

$$(x, y) \sim (x', y') :\Leftrightarrow y = y'. \tag{1.12}$$

Then, the quotient space $\mathbb{R}^2/\sim$ is a quotient space of $\mathbb{R}^2$.

Remark. Note that quotient set is not the subset. Therefore, quotient space is not the subspace.

2 Basis and Dimension

I wish now you are familiar with the operation on the vector space and its importance. For simplicity of notation, I will skip s-multiplication symbol $\heartsuit$ from now. That is for the $\mathbb{K}$-vector space $(V, +, \heartsuit)$:

$$\forall\, a \in \mathbb{K} : v \in V : \quad av \;:\Leftrightarrow\; a \heartsuit v. \tag{2.1}$$

Since the addition and s-multiplication are closed, we can apply them iteratively. We can, particularly, define the addition of multiple of given vectors.

Definition 2.1 (Linear Combination)**.** Let V be a $\mathbb{K}$-vector space. A *linear combination* of $v_1, v_2, \ldots, v_n \in V$ is:

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \in V, \tag{2.2}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{K}$.

Note that the vector space is important to tell what is vector. In fact, we can define a subspace of V using linear combinations of given vectors.

Box 2.1: Span

Let V be a $\mathbb{K}$-vector space. The set of all possible linear combination of given $v_1, v_2, \ldots, v_n \in V$, i.e.,

$$\operatorname{span}(v_1, \ldots, v_n) \coloneqq \{a_1 v_1 + \ldots + a_n v_n \in V \mid a_1, \ldots a_n \in \mathbb{K}\} \tag{2.3}$$

is called the *span of* $v_1, \ldots, v_n$. If $\operatorname{span}(v_1, \ldots, v_n) = U$ for some subspace U, we call $v_1, \ldots, v_n$ *spans* U. For the special case, $\operatorname{span}(v_1, \ldots, v_n) = V$, we call $v_1, \ldots, v_n$ *spans* V. The set of vectors $\{v_1, \ldots, v_n\}$ is called the *spanning set* or *generating set*.

Remark. Note that the span space can be obtained just by extending the set to make the operation closed for given vectors. Therefore, it is the smallest subspace of V which contains all vectors $v_1, \ldots, v_n$.

We can ask the reverse question. If $v_1, \ldots, v_n$ *spans* U, are the given vectors minimal? Or do we need all $v_1, \ldots, v_n$? We can answer by checking independence/dependence of given vectors.

Definition 2.2 (Linearly Independent)**.** Let V be a $\mathbb{K}$-vector space. We call vectors $v_1, v_2, \ldots, v_n \in V$ are *linearly independent* when

$$a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0 \iff a_1 = a_2 = \ldots = a_n = 0. \tag{2.4}$$

We can also write the same definition using the subset by just choosing $\{v_1, v_2, \ldots, v_n\} \subseteq V$. In this case, we also say that the set $\{v_1, v_2, \ldots, v_n\}$ is the set of linearly independent vectors. Recall that $\varnothing \subseteq V$, therefore, we can also think about the independence of no vector. Recall "Ex Falso quodlibet", if there is no given vector, it is linearly independent. If the given vectors are not linearly independent, we call they are *linearly dependent*.

Theorem 2.1. If $v_1, v_2, \ldots, v_n \in V$ are linearly dependent. Then,

$$\exists\, k \in \{1, 2, \ldots, n\} : v_k = \sum_{i \neq k} a_i v_i. \tag{2.5}$$

In other words,

$$\operatorname{span}(v_1, \ldots, v_n) = \operatorname{span}(v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n) \tag{2.6}$$

Definition 2.3 (Dimension)**.** The *dimension* of a vector space is the number of possible set of independent vectors in it or cardinality of independent vector set in it. For $\mathbb{K}$-vector space V, we denote $\dim V$ for dimension of V. If the dimension of V is finite, we write $\dim V < \infty$.

Theorem 2.2. If the dimension of vector space is *finite*, we can define it also using the spanning set. More precisely, let V be a $\mathbb{K}$-vector space. When $\operatorname{span}(v_1, \ldots, v_n) = U$, and $|\operatorname{span}(v_1, \ldots, v_n)| = n$, the dimension of U is n. We write $\dim V = n$.

We do not want to suffer from the convergence of series. It needs knowledge of analysis or topology. Therefore, we will talk about only the linear combination of finite vectors.

Box 2.2: Algebraic Basis (or Hamel Basis)

Let V be a $\mathbb{K}$-vector space. A subset $B \subseteq V$ is called a *Hamel basis* or *algebraic basis* for V if its elements are linearly independent and its finite subset $\{e_1, \ldots, e_n\} \subseteq B$ spans V, that is

$$\forall v \in V : \exists!\, v_1, \ldots, v_n \in \mathbb{K} : v = \sum_{i=1}^{n} v_n e_n. \tag{2.7}$$

The existence of Hamel basis is guaranteed by the axiom of choice. The numbers $v_1, \ldots, v_n$ are called the *components*

Remark. In physics, we say B is *complete* when $\{e_1, \ldots, e_n\} \subseteq B$ spans whole space V.

Remark. The basis is not unique. But all Hamel basis for the same space have same cardinality.

Theorem 2.3. Let V be a $\mathbb{K}$-vector space. The *dimension* of V is $\dim V = |B|$, where B is a Hamel basis for V.

Remark. Do not confuse with the linearly independence and nonuniqueness of basis. When we chose Hamel basis, the components for the linear expansion is unique.

Theorem 2.4. Let V be a vector space and B a Hamel basis of V. Then B is a minimal spanning and maximal independent subset of V, i.e., if $S \subseteq V$, then

- $\mathrm{span}(S) = V \;\Rightarrow\; |S| \geq |B|$;
- S is linearly independent $\;\Rightarrow\; |S| \leq |B|$.

Theorem 2.5. For finite dimensional vector space, $\dim V < \infty$ and $S \subseteq V$,

- if $\mathrm{span}(S) = V$ and $|S| = \dim V$, then S is a Hamel basis of V;
- if S is linearly independent and $|S| = \dim V$, then S is a Hamel basis of V.

Example 2.5.1. Let U and W be a subspaces for $\mathbb{K}$-vector space V. Moreover, let $\{u_1, \ldots, u_n\}$ and $\{w_1, \ldots, w_m\}$ be basis of each subspace. Then,

$$\mathrm{span}(u_1, \ldots, u_n, w_1, \ldots, w_m) = U + W. \tag{2.8}$$

Particularly, if $U \cap W = \{0\}$, $\{u_1, \ldots, u_n, w_1, \ldots, w_m\}$ is the basis of $U \oplus W$.

Theorem 2.6. Let U and W be a subspaces for $\mathbb{K}$-vector space V. Then,

$$\dim\,(V + W) = \dim V + \dim W - \dim\,(V \cap W). \tag{2.9}$$

Particularly, for $V \cap U = \{0\}$,

$$\dim\,(V \oplus W) = \dim V + \dim W \tag{2.10}$$

One may recall that it was useful to write every vectors with basis expansion or components. But, conceptually, it is not good and make more confusion for understanding. There is famous saying *"A gentleman never chooses a basis."* I strongly recommend the readers not to use the basis expansion as much as possible especially they want to understand the concepts in linear algebra.

Note that I do not introduce any other operation such as inner product. Therefore, you do not have the mechanism to find coefficients for given vector.

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