Inner Product Space

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Abstract: References are of course my favorite [1, 2] and additionally, [3–5]. For the basis, I think [6] explains very well.

Contents

1 Bilinear Form and Sesquilinear Form

We have learnt the definition of linear map. It was a unary map. We can extend the map for n-ary maps using the Cartesian products.

Box 1.1: Multilinear Map

Let $(V_1, +_{V_1}, \heartsuit_{V_1}), \ldots, (V_n, +_{V_n}, \heartsuit_{V_n})$, and $(W, +_W, \heartsuit_W)$ be $\mathbb{K}$-vector spaces. If the map $\phi : V_1 \times \ldots \times V_n \to W$ is linear in each slot. That is, it satisfies $\forall\, a \in \mathbb{K} : \forall\, i \in \{1, \ldots, \} : \forall\, u_i v_i, \in V_i$

$$\phi(a\heartsuit_{V_1} u_1 +_{V_1} v_1, v_2, \ldots, v_n) = a\heartsuit_W \phi(u_1, v_2, \ldots, v_n) + \phi(v_1, \ldots, v_n) \tag{1.1}$$

$$\vdots \tag{1.2}$$

$$\phi(v_1, v_2, \ldots, a\heartsuit_{V_n} u_n +_{V_n} v_n) = a\heartsuit_W \phi(v_1, v_2, \ldots, u_n) + \phi(v_1, \ldots, v_n) \tag{1.3}$$

Remark. The multilinear map is one of the most general map we can think in linear algebra, since it is a map between vector spaces. In physics, we usually want to calculate vectors in the same space. In other words, we want the domain to be the Cartesian product of the same space. Many concepts in the linear algebra such as inner product (in Euclidean space), outer product, tensors are all just some special case of linear maps.

Definition 1.1 (Multilinear Form)**.** Let $(V, +_V, \heartsuit_V)$ be $\mathbb{K}$-vector spaces. If the map

$$\phi : V^n \to \mathbb{K} \tag{1.4}$$

is multilinear, it is called the *multilinear n-form* or *(covariant) n-tensor*.

Remark. The multilinear 2-form is simply called a bilinear form.

Example 1.0.1 (Integration)**.** For the vector space of real functions, we can define bilinear form as definite integration of the product of real functions, e.g.,

$$I : V^2 \to \mathbb{K} \quad \text{by} \quad (f, g) \mapsto \int_0^1 f(x)g(x)\mathrm{d}x. \tag{1.5}$$

Note that the multilinear function is not a linear function.

Definition 1.2 (quadratic form)**.** Let $\phi : V^2 \to \mathbb{K}$ be a bilinear form. A function $q : V \to \mathbb{K}$ defined

$$\forall\, v \in V : q(v) \coloneqq \phi(v, v) \tag{1.6}$$

is called a *quadratic form*.

Remark. What if we put only one vector in the slot? In that case, the result of bilinear map is a scalar and we need only one vector for it. Therefore, it is just a one-form.

Definition 1.3 (Slot Notation)**.** For given bilinear form $\phi : V^2 \to \mathbb{K}$, we can find a one-form by putting some vector $v \in V$ for it. That is

$$(\phi_v^L : V \to \mathbb{K}) \in V^\vee \qquad \text{such that} \qquad \forall\, w \in V : \phi_v^L(w) \coloneqq \phi(v, w); \tag{1.7}$$

$$(\phi_v^R : V \to \mathbb{K}) \in V^\vee \qquad \text{such that} \qquad \forall\, w \in V : \phi_v^R(w) \coloneqq \phi(w, v). \tag{1.8}$$

We usually use the notation $\phi(v, \cdot)$ for ϕ_v^L and $\phi(\cdot, v)$ for ϕ_v^R.

Remark. Recall that one-form is also a linear map.

Definition 1.4 (Left/Right Radicals)**.** The *left/right radicals* are the intersection of all kernels of $\phi_V^{L/R}$. That is

$$\mathrm{rad}_L(\phi) \coloneqq \bigcap_{v \in V} \ker \phi_v^L = \{w \in V \mid \forall\, v \in V : \phi(v, w) = 0\}; \tag{1.9}$$

$$\mathrm{rad}_R(\phi) \coloneqq \bigcap_{v \in V} \ker \phi_v^R = \{w \in V \mid \forall\, v \in V : \phi(w, v) = 0\}; \tag{1.10}$$

Remark. The radical is just the set of vectors whose value of bilinear form is always zero for any input in other slot.

Definition 1.5 (Nondegenerate)**.** A bilinear form $\phi : V^2 \to \mathbb{K}$ is called *nondegenerate* iff

$$\mathrm{rad}_L(\phi) = \mathrm{rad}_R(\phi) = \{0\}. \tag{1.11}$$

where $0 \in V$ is the zero vector in V. Otherwise, it is called *degenerate*.

We do not yet tell the relation between two slots (inputs) today, we will concentrate on one simple case.

Box 1.2: Symmetric Bilinear form

Let $\phi : V^2 \to \mathbb{K}$ be a bilinear form. If it satisfies

$$\forall\, v_1, v_2 \in V : \quad \phi(v_1, v_2) = \phi(v_2, v_1), \tag{1.12}$$

it is called *symmetric*.

Example 1.0.2 (zero map)**.** A zero map $\mathfrak{o} : V^2 \to \mathbb{K}$ by $(v_1, v_2) \mapsto 0$ is a symmetric bilinear form.

Example 1.0.3 (four-vector product)**.** In special relativity, we learnt that the four-vector which can be represented with one time coordinate and three spatial coordinates, $v = (v^0, v^1, v^2, v^3)$. The four-vector product $\cdot$

$$v \cdot w \coloneqq -v^0 w^0 + \sum_{i=1}^{3} v^i w^i \tag{1.13}$$

is a symmetric bilinear form.

Remark. The four-vector product we have shown is usually called *dot product* or *inner product* of Minkowski space, or *Lorentzian inner product.* But this *bona fide* inner product is not an inner product.

Definition 1.6 (dot product, inner product, scalar product in real vector space)**.** Let V be a real vector space. An *inner product* $\cdot : V^2 \to \mathbb{R}$ is a nondegenerate positive definite symmetric bilinear 2-form. The positive and definite means:

- **positive:** $\forall v \in V : v \cdot v \geq 0$;
- **definite:** $\forall v \in V : (v \cdot v = 0 \iff v = 0)$.

Note that inner product is an extra structure, some special bilinear form in the vector space. We do not need them for defining the vector space. On the other hand, because it is so simple and intuitive, we want to extend it more general case, that is, to the complex vector space. It is, however, not possible to define positive and definiteness in bilinear map. Since, if $v \cdot v > 0$,

$$(iv) \cdot (iv) = i^2 (v \cdot v) = -(v \cdot v) < 0. \tag{1.14}$$

Therefore, we want new map, which reduces to the bilinear form if we restrict the field to be a real number. It is called a sesquilinear form[1]

Definition 1.7 (Sesquilinear form)**.** Let $(V, +_V, \heartsuit_v)$ be a $\mathbb{K}$-vector space. A *sesquilinear form* is a bilinear map $\langle\ ,\ \rangle : V^2 \to \mathbb{K}$ which is

- **linear on the right slot:**

$$\forall\, v, w_1, w_2 \in V : \forall\, z \in \mathbb{C} : \langle v, w_1 +_V z \heartsuit_V w_2 \rangle = \langle v, w_1 \rangle + z \langle v, w_2 \rangle; \tag{1.15}$$

- **antilinear on the left slot:**

$$\forall\, v_1, v_2, w \in V : \forall\, z \in \mathbb{C} : \langle v_1 +_V z \heartsuit_V v_2, w \rangle = \langle v_1, w \rangle + \bar{z} \langle v_2, w \rangle \tag{1.16}$$

  where $\bar{z}$ is the complex conjugate.[2]

Remark. If $\mathbb{K} = \mathbb{R}$, the sesquilinear form is just the bilinear form.

Definition 1.8 (Symmetric)**.** A sesquilinear map $\langle\ ,\ \rangle : V^2 \to \mathbb{K}$ is *symmetric* if

$$\forall\, v, w \in V : \quad \langle v, w \rangle = \overline{\langle w, v \rangle} \tag{1.17}$$

Box 1.3: Hermitian Form

For complex vector field V, the symmetric sesquilinear form is called the *Hermitian form.*

[1]sesqui is a Latin word for one and half times.

[2]In fact, is is some 'conjugation' defined with involution. But, since we want to focus on the real and complex vector space, we will just use the complex conjugate here.

2 Inner Product and Normed Vector Space

Now we are ready to introduce the inner product in general.

Box 2.1: Inner Product

Let $(V, +, \heartsuit)$ be a $\mathbb{K}$-vector space. An *inner product* $\langle\ ,\ \rangle : V^2 \to \mathbb{K}$ is a positive definite Hermitian form. That is

- **positive:** $\forall\, v \in V : \langle v, v\rangle \geq 0$;
- **definite:** $\forall\, v \in V : \langle v, v\rangle = 0 \iff v = 0$;
- **symmetric:** $\forall\, v, w \in V : \langle w, v\rangle = \overline{\langle v, w\rangle}$;
- **sesquilinear:** $\forall\, z \in \mathbb{K} : \forall v, w_1, w_2 \in V : \langle v, w_1 + z w_2\rangle = \langle v, w_1\rangle + z\langle v, w_2\rangle$.

Note that because of symmetry condition, we do not need to check left slot. The vector space with inner product is $\big((V, +, \heartsuit), \langle\ ,\ \rangle\big)$ is called an *inner product space*. In general, the basic operations for the vector space $+, \heartsuit$ are omitted and we just write pair $(V, \langle\ ,\ \rangle)$.

Remark. When we learnt the dual map, the coordinate map gives us the coefficients in the linear combination expression of given basis. On the other hand, we do not know the relation between basis vectors, because we need binary map to get the relation of two vectors. This strong additional map, inner product, finally allow us to settle the relation between basis vectors such as orthogonality.

Definition 2.1 (orthogonal)**.** Two vectors $v, w \in V$ are called *orthogonal* if $\langle v, w\rangle = 0$

Theorem 2.1 (orthogonality of 0)**.** For $0 \in V$, $\forall v \in V : \langle 0, v\rangle = 0$. Therefore 0 is orthogonal to every vector even for itself.

For the basis vector, we also want it not only orthogonal, but also normalised. For it, we need the definition of norm.

Box 2.2: Norm

Let V be a $\mathbb{K}$-vector space. The *norm* on V, denoted $\|\cdot\| : V \to \mathbb{R}$, satisfies following four axioms:

- **positive:** $\forall\, v \in V : \|v\| \geq 0$;
- **definite:** $\forall\, v \in V : \|v\| = 0 \iff v = 0$;
- **absolute homogeneous:** $\forall\, z \in \mathbb{K} : v \in V : \|zv\| = |z|\|v\|$;
- **Triangle Inequality:** $\forall\, v, w \in V : \|v + w\| \leq \|v\| + \|w\|$.

A vector space equipped with norm, denoted as a pair $(V, \|\cdot\|)$, is called a normed vector space.

Remark. The inner product and norm has the common property of positive definiteness. Therefore, we can naturally define the norm induced from the inner product.[3]

Theorem 2.2. We induce a *norm* in the inner product space defined by

$$\|v\| \coloneqq \sqrt{\langle v, v\rangle}. \tag{2.1}$$

We will prove this after introducing several theorems. Note that the following theorems can be written with inner products only. But then, we do not have the meaning of norm (length). It is logical to write all with only inner products and then introduce norm expression later. But I do not want to write the same things again and again. Forgive me for the laziness.

Theorem 2.3 (Pythagorean Theorem)**.** Let $u, v \in V$ be orthogonal, i.e., $\langle u, v\rangle = 0$. Then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2. \tag{2.2}$$

Proof.

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v\rangle \\ &= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\ &= \langle u, u\rangle + \langle v, v\rangle \\ &= \|u\|^2 + \|v\|^2. \end{aligned}$$

□

For given two vector, we can always express one vector in the linear combination of the parallel vector and the orthogonal vector to the other vector.

Theorem 2.4 (Orthogonal Decomposition)**.** Let $u, v \in V$ and $v \neq 0$. Then we can write $u = \frac{\langle u,v\rangle}{\|v\|^2} v + w$ where

$$w \coloneqq u - \frac{\langle u, v\rangle}{\|v\|^2} v. \tag{2.3}$$

which is orthogonal to v, i.e., $\langle w, v\rangle = 0$

Lemma 2.5 (Cauchy-Schwarz Inequality)**.**

$$\forall u, v \in V : \quad |\langle u, v\rangle| \leq \|u\|\|v\|. \tag{2.4}$$

The equality holds if $\exists c \in \mathbb{K} : u = cv$.

[3] It is the most convenient and popular choice, but it is not necessary. One can define other norm also. In fact, inner product space is one of the special case of normed vector space.

Proof. We use the orthogonal decomposition of u with v.

$$\begin{aligned}\|u\|^2 &= \left\| \frac{\langle u, v\rangle}{\|v\|^2} v + w \right\|^2 \\ &= \left\| \frac{\langle u, v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 \\ &= \left| \frac{\langle u, v\rangle}{\|v\|^2} \right|^2 \|v\|^2 + \|w\|^2 \\ &= \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \|w\|^2.\end{aligned}$$

In the second line, we use the Pythagorean theorem since, they are orthogonal. Multiplying $\|v\|^2$ on both sides gives

$$\|u\|^2\|v\|^2 = |\langle u, v\rangle|^2 + \|v\|^2\|w\|^2 \geq |\langle u, v\rangle|^2 \tag{2.5}$$

Equality holds when $w = 0$. □

Proof. Finally, we can now prove that the induced norm defined above, i.e., $\|v\| := \sqrt{\langle v, v\rangle}$ is really a norm. It is enough to check that this definition satisfies the triangle inequality holds.

$$\begin{aligned}\|v + w\|^2 &= \langle v + w, v + w\rangle \\ &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= \|v\|^2 + 2\Re[\langle v, w\rangle] + \|w\|^2 \\ &\leq \|v\|^2 + \|w\|^2 + 2|\langle v, w\rangle| \\ &\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| \\ &= (\|v\| + \|w\|)^2\end{aligned}$$

In the fifth line, the Cauchy-Schwarz inequality is used. □

We have two different but seems related spaces, inner product spaces and normed spaces. In fact, inner product space is one of the special case of normed vector space. So in the logical flow of our lecture, general to more specific concept, we might want to define inner product with norm.

Remark. In fact, one sometimes define inner products with norm. It is polarisation identity. In real vector space,

$$\langle u, v\rangle := \frac{1}{4}\big(\|u + v\| - \|u - v\|\big). \tag{2.6}$$

In complex vector space, we can do the same but we need to see real and imaginary part separately:

$$\langle u, v\rangle := \frac{1}{4} i^k \|u + i^{4-k} v\|^2 = \frac{1}{4}\big[\big(\|u + v\|^2 - \|u - v\|^2\big) + i\big(\|u - iv\|^2 - \|u + iv\|^2\big)\big]. \tag{2.7}$$

If we accept the definition of induced norm $\|v\| \coloneqq \sqrt{\langle v, v\rangle}$, one can show that the polarisation identity holds. But, we want to show really it can be used for the definition of inner product. Therefore, we need to show it is true without the definition of induced norm. In fact, there is a theorem for it.

Theorem 2.6 (Jordan-von Neumann theorem [7])**.** Let $(V, \|\cdot\|)$ be a complex normed vector space. Then there is a unique inner product $\|v\| \coloneqq \sqrt{\langle v, v\rangle}$ on V iff it satisfies the parallelogram identity:

$$\forall\, u, v \in V: \quad \|u+v\|^2 + \|u-v\|^2 = 2\big(\|u\|^2 + \|v\|^2\big). \tag{2.8}$$

Proof. ($\Longrightarrow$)

$$\begin{aligned}\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle + \big(\langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle\big) \\ &= 2\langle u,u\rangle + 2\langle v,v\rangle \\ &= 2\big(\|u\|^2 + \|v\|^2\big).\end{aligned}$$

($\Longleftarrow$) For the proof, we will use the polarisation identity. That is we will define the inner product as

$$\langle u, v\rangle \coloneqq \frac{1}{4} i^k \|u + i^{4-k} v\|^2 = \frac{1}{4}\big[\big(\|u+v\|^2 - \|u-v\|^2\big) + i\big(\|u-iv\|^2 - \|u+iv\|^2\big). \tag{2.9}$$

Now we need to show that it is really an inner product. Recall that inner product is positive definite Hermitian (symmetric sesquilinear) form. Let us show each one separately.

- *positive definite*:

$$\begin{aligned}\langle v, v\rangle \coloneqq& \frac{1}{4}\big[\big(\|v+v\|^2 - \|v-v\|^2\big) + i\big(\|v-iv\|^2 - \|v+iv\|^2\big)\big] \\ =& \frac{1}{4}\big[\big(\|2v\|^2 - \|0\|^2\big) + i\big(\|v-iv\|^2 - \|v+iv\|^2\big)\big] \\ =& \frac{1}{4}\big[4\big(\|v\|^2 - \|0\|^2\big) + i\big(2\|v\|^2 - 2\|v\|^2\big)\big] \\ =& \|v\|^2\end{aligned}$$

  where in the third line, we used the absolute homogeneity. Since the norm is positive definite, inner product is also positive definite.

- *symmetric*: We can show by direct calculation.

$$\begin{aligned}\overline{\langle u, v\rangle} \coloneqq& \frac{1}{4}\big[\big(\|u+v\|^2 - \|u-v\|^2\big) - i\big(\|u-iv\|^2 - \|u+iv\|^2\big)\big] \\ =& \frac{1}{4}\big[\big(\|v+u\|^2 - \|-(v-u)\|^2\big) - i\big(\|-i(v+iu)\|^2 - \|i(v-iu)\|^2\big)\big] \\ =& \frac{1}{4}\big[\big(\|v+u\|^2 - \|v-u\|^2\big) - i\big(\|v+iu\|^2 - \|v-iu\|^2\big)\big] \\ =& \frac{1}{4}\big[\big(\|v+u\|^2 - \|v-u\|^2\big) + i\big(\|v-iu\|^2 - \|v+iu\|^2\big)\big] \\ &= \langle v, u\rangle\end{aligned}$$

  where absolute homogeniety used in the third line.

- *sesquilinear:* This is little bit complicated. Since we show it is symmetric, we only need to show the linearity in the right slot.
  - *additivity:* The paralleogram identity is used here (third and fifth identity in the following).

$$\begin{aligned}
\langle u, v+w\rangle &= \frac{1}{4}\big[\big(\|u+v+w\|^2 - \|u-(v+w)\|^2\big) + i\big(\|u-i(v+w)\|^2 - \|u+i(v+w)\|^2\big)\big] \\
&= \frac{1}{4}\big[\big(\|u+v+w\|^2 + \|u+v-w\|^2 - \|u+v-w\|^2 - \|u-v-w\|^2\big) \\
&\quad + i\big(\|u-iv-iw\|^2 + \|u-iv+iw\|^2 - \|u-iv+iw\|^2 - \|u+iv+iw\|^2\big) \\
&= \frac{1}{2}\big[\big(\|u+v\|^2 + \|w\|^2 - \|u-w\|^2 - \|v\|^2\big) \\
&\quad + i\big(\|u-iv\|^2 + \|iw\|^2 - \|u+iw\|^2 - \|iv\|^2\big) \\
&= \frac{1}{2}\big[\big(\|u+v\|^2 + \|u\|^2 + \|w\|^2 - \|u-w\|^2 - \|u\|^2 - \|v\|^2\big) \\
&\quad + i\big(\|u-iv\|^2 + \|u\|^2 + \|iw\|^2 - \|u+iw\|^2 - \|u\|^2 - \|iv\|^2\big) \\
&= \frac{1}{4}\big[\big(\|u+v\|^2 + \|u-v\|^2 + \|u+w\|^2 - \|u-w\|^2\big) \\
&\quad + i\big(\|u-iv\|^2 + \|u+iv\|^2 + \|u-iw\|^2 - \|u+iw\|^2\big) \\
&= \langle u, v\rangle + \langle u, w\rangle.
\end{aligned}$$

  - *homogeneity:* Showing homogeneity is much more subtle than additivity. We can start with additivity and extending the domain of scalar. Let us start with simplest case, the following can be obtained by direct calculation:

$$\forall\, z \in \{0, 1\} : \langle u, zv\rangle = z\langle u, v\rangle. \tag{2.10}$$

    Then, using the additivity iteratively, we can extend the domain of scalar to real numbers.

$$\forall\, z \in \mathbb{Z} : \langle u, zv\rangle = z\langle u, v\rangle. \tag{2.11}$$

    Moreover, since the s-multiplication is closed, if $v \in V \implies z^{-1}v \in V$. Therefore, we can rewrite the same equation in the case of $z^{-1}v$

$$\forall\, z \in \mathbb{Z} : z\langle u, \frac{v}{z}\rangle = \langle u, z\frac{v}{z}\rangle = \langle u, v\rangle. \tag{2.12}$$

    Dividing both side with z gives

$$\forall\, z \in \mathbb{Z} : \langle u, \frac{v}{z}\rangle = \frac{1}{z}\langle u, v\rangle. \tag{2.13}$$

    Using addtivity again we can extend to the scalars which are homogeneous to whole rational numbers:

$$\forall\, z \in \mathbb{Q} : \langle u, zv\rangle = z\langle u, v\rangle. \tag{2.14}$$

For the rigorous proof to go to the real numbers, we need the knowledge of analysis. I will just comment here that for any real number, you can find enough close rational number to it[4]. So, homogeneity holds in real numbers. Finally,

$$\begin{aligned}\langle u, iv\rangle &= \frac{1}{4}\big[\big(\|u+iv\|^2 - \|u-iv\|^2\big) + i\big(\|u+v\|^2 - \|u-v\|^2\big)\big] \\ &= \frac{1}{4}i\big[-i\big(\|u+iv\|^2 - \|u-iv\|^2\big) + \big(\|u+v\|^2 - \|u-v\|^2\big)\big] \\ &= i\langle u, v\rangle.\end{aligned}$$

Since

$$\forall\, z \in \mathbb{C} : \exists\, a, b \in \mathbb{R} : \quad z = a + ib, \tag{2.15}$$

we finally conclude that

$$\forall z \in \mathbb{Q} : \langle u, zv\rangle = z\langle u, v\rangle. \tag{2.16}$$

□

3 Orthonormal Basis

We are now ready to define the orthonormality.

Box 3.1: Orhonormal basis

Let $(V, \langle\ ,\ \rangle)$ be an inner product space. Then the set of vectors $B = \{e_1, \dots, e_n\} \subseteq V$ is called *orthonormal* if it is

- **orthogonal**: $\forall e_i, e_j \in B : (e_i \neq e_j \iff \langle e_i, e_j\rangle = 0)$;
- **normalised**: $\forall e_i \in B : \langle e_i, e_i\rangle = 1$.

The two conditions can be written in a simple form using the kronecker delta symbol:

$$\forall e_i, e_j \in B : \quad \langle e_i, e_j\rangle = \delta_{ij}. \tag{3.1}$$

If B is a basis for V, B is called an *orthonormal basis*.

Theorem 3.1 (Bessel's inequality)**.** If the set $B = \{e_1, \dots, e_n\}$ is an orthonormal set in V,

$$|\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_n\rangle|^2 \leq \|v\|^2. \tag{3.2}$$

The equality holds when B is a basis of V.

Recall that the basis can be expressed in the linear combination of basis vectors. Using the inner product, for the n-dimensional vector space V,

$$\forall v \in V : v = \sum_{i=1}^{n} \langle e_i, v\rangle e_i. \tag{3.3}$$

[4]This property is called that "rational numbers form a dense subset of real numbers.'

Recall that the components of vectors are obtained by applying dual basis to the vector.

Box 3.2: Component

Let $(V, \langle\ ,\ \rangle)$ be an inner product space. Then the set of vectors $B = \{e_1, \ldots, e_n\} \subseteq V$ be an orthonormal basis. And let $\{\alpha^1, \ldots, \alpha^n\}$ be the dual basis of B. Then,

$$\forall v \in V : \qquad v^i := \alpha^i(v) = \langle e_i, v \rangle. \tag{3.4}$$

The number $v^i \in \mathbb{K}$ is called the components of v. In other words, using slot notation, we can identify inner product with dual basis:

$$\langle e_i, \cdot \rangle \quad :\Leftrightarrow \quad \alpha^i. \tag{3.5}$$

We already know how convenient the component is. Note that the components depend on the basis. When the basis is known, then one can obtain the components.

The Gram-Schmidt procedure is the way to obtain the orthonormal vectors from the given linearly independent vectors.

Box 3.3: Gram-Schmidt Procedure

The Gram-Schmidt procedure is the way of finding the orthonormal vectors from given linearly independent vectors. Let the vectors $v_1, \ldots, v_m \in V$ be linearly independent. Then we do the following procedure:

1. Choose one vector. For example, let us assume that we choose v_1.
2. Define new vector by normalising it:
$$e_1 := \frac{1}{\|v_1\|} v_1; \tag{3.6}$$
3. Choose another vector. For example, v_2, and make orthogonal decomposition with e_1:
$$v_2 = \langle e_1, v_2 \rangle e_1 + v_2'. \tag{3.7}$$
4. Define new vector by normalising v_2':
$$e_2 := \frac{1}{\|v_2'\|} v_2' \tag{3.8}$$
5. Do the similar procedure iteratively:
$$v_3 = \langle e_1, v_3 \rangle e_1 + \langle e_2, v_3 \rangle e_2 + v_3', \tag{3.9}$$
$$e_3 := \frac{1}{\|v_3'\|} v_3' \tag{3.10}$$
$$\vdots \tag{3.11}$$

As a result, we have set of orthonormal vectors $\{e_1, \ldots, e_n\}$.

Remark. Now, we can use the usual expression, such as column matrix for the vector, and matrix for the operator and so on.

4 Matrices

Now let us review the idea of using set of numbers to express vector. Please keep in mind that this is *representation*. We can talk more on linear algebra without saying this expressions. But, I think it is good time to review here to grasp the familiarity of our younger time linear algebra and to relate a concrete calculation. But I also hope you feel uncomfortable in this expressions.

In this section, we will assume V, W is $\mathbb{K}$-vector space whose dimension is n, m each. We will assume we have chosen the basis $\{e_1, \ldots, e_n\}$ as a basis of V and $\{f_1, \ldots, f_m\}$ for the basis of W.

4.1 Basic Expression

Recall that using the basis, we can express a given vector with *unique* linear combination

$$\forall\, v \in V : \exists! v^1, \ldots, v^n \in \mathbb{K} : v = \sum_{i=1}^{n} v^i e_i \tag{4.1}$$

Definition 4.1 (Column vector)**.** The vector $v = v^1 e_1 + \ldots v^n e_n \in V$ is expressed

$$\mathbf{v} = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}. \tag{4.2}$$

The expression is called *column vector*.

Remark. Note that the all the information of vectors are in each numbers (components). The shape of the number is not important. You can use any form of expression. The only importance is you need express ordered pair of numbers. Just n-tuple of numbers is enough to express the vectors. The column vector is just one traditional expression of vectors.

One can do the similar expression to the vector space W. Now let us express the linear map $A \in \mathrm{Hom}(V, W)$. Recall that the set of linear maps also form a vector space and $\dim\big(\mathrm{Hom}(V, W)\big) = m \times n$. Therefore one can use the column matrix expression

$$\mathbf{A} = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^{m\times n} \end{pmatrix}. \tag{4.3}$$

It is, however, not that satisfactory expression, because it hides the role of map (relating unique vector to the given vector in domain). The linear map acts

$$w = \sum_{j=1}^{m} w^j f_j = Av = A\Big(\sum_i v^i e_i\Big) = \sum_i v^i (Ae_i). \tag{4.4}$$

In the last equality, we used the linearity of linear map. Note that even though it uses the same notation, the addition is defined in the different vector space:

$$\sum_i v^i e_i = v^1 e_1 +_V \ldots +_V v^n e_n \longrightarrow \sum_i v^i (Ae_i) = v^1(Ae_1) +_W \ldots +_W v^n(Ae_n). \qquad (4.5)$$

Recall that all the information of w is in m-numbers w^j and all the information in v is in the n-numbers v^i. The role of A is relating w^j to v^i. Therefore, it is good to have m contribution and n contribution separately. Let us choose $v^j = \delta^j_i$ for some $i \in \{1, \ldots, m\}$ and the resulting vector A_i. Then we can rewrite above equation by[5]

$$A_i = \sum_j A^j{}_i f_j = Ae_i \qquad (4.6)$$

Note the different role of i and j. With this, we can rewrite the action of A on general vector v:

$$Av = A\Big(\sum_i v^i e_i\Big) = \sum_i v^i (Ae_i) = \sum_i v^i \Big(\sum_j A^j{}_i f_j\Big) = \sum_j \Big(\sum_i A^j{}_i v^i\Big) f_j. \qquad (4.7)$$

In other words, if we identify $w = Av$, the components have the relation

$$w^j = A^j{}_i v^j. \qquad (4.8)$$

Based on this, we define the matrix representation of the linear map.

Definition 4.2 (Matrix expression of linear map)**.** Let $A \in \mathrm{Hom}\,(V, W)$. We express

$$\mathbf{A} = \begin{pmatrix} A^1{}_1 & A^1{}_2 & \ldots & A^1{}_n \\ A^2{}_1 & A^2{}_2 & \ldots & A^2{}_n \\ \vdots & \vdots & \ddots & \vdots \\ A^m{}_1 & A^m{}_2 & \ldots & A^m{}_n \end{pmatrix}. \qquad (4.9)$$

This is called *matrix* representation of A.

The matrix is good representation of linear map. We will define the operations (algebras) of it to be consistent with the action of the linear map.

Definition 4.3 (Matrix multiplication on column matrix)**.** Let $A \in \mathrm{Hom}\,(V, W)$. All the information of the equation $w = Av$ are in the components.

$$w^j = A^j{}_i v^i. \qquad (4.10)$$

The matrix multiplication on column vector is *defined* to satisfy it:

$$\mathbf{Av} = \begin{pmatrix} A^1{}_1 & A^1{}_2 & \ldots & A^1{}_n \\ A^2{}_1 & A^2{}_2 & \ldots & A^2{}_n \\ \vdots & \vdots & \ddots & \vdots \\ A^m{}_1 & A^m{}_2 & \ldots & A^m{}_n \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} = \begin{pmatrix} \sum_i A^1{}_i v^i \\ \sum_i A^2{}_i v^i \\ \vdots \\ \sum_i A^m{}_i v^i \end{pmatrix}. \qquad (4.11)$$

[5] Be careful on the notation. A is a linear map, A_i is a vector and $A^j{}_i$ is just a number (component of A_i).

Remark. The matrix multiplication is defined to express an composition (iterative action) of linear maps. For example, Let there is also a $\mathbb{K}$-vector space U and $B \in \mathrm{Hom}(U, V), A \in \mathrm{Hom}(V, W)$. Then, we can define the composition of the map

$$(A \circ B) : U \to W \qquad \text{by } u \mapsto (A \circ B)u = A\big(Bu\big). \tag{4.12}$$

One can check easily that the information of $w = (A \circ B)u$ is in components equation

$$w^i = \sum_k \Big(\sum_j A^i{}_j B^j{}_k \Big) u^k \tag{4.13}$$

Definition 4.4 (Multiplication between matrices)**.** The matrix multiplication is defined to express an composition (iterative action) of linear maps. For example, Let there is also a $\mathbb{K}$-vector space U and $B \in \mathrm{Hom}(U, V), A \in \mathrm{Hom}(V, W)$. The matrix multiplication is defined to satisfy

$$w^i = \sum_k \Big(\sum_j A^i{}_j B^j{}_k \Big) u^k. \tag{4.14}$$

That is:

$$\mathbf{AB} = \begin{pmatrix} A^1{}_1 & A^1{}_2 & \dots & A^1{}_n \\ A^2{}_1 & A^2{}_2 & \dots & A^2{}_n \\ \vdots & \vdots & \ddots & \vdots \\ A^m{}_1 & A^m{}_2 & \dots & A^m{}_n \end{pmatrix} \begin{pmatrix} B^1{}_1 & B^1{}_2 & \dots & B^1{}_{\dim U} \\ B^2{}_1 & B^2{}_2 & \dots & B^2{}_{\dim U} \\ \vdots & \vdots & \ddots & \vdots \\ B^n{}_1 & B^m{}_2 & \dots & B^n{}_{\dim U}. \end{pmatrix} \tag{4.15}$$

Remark. You might heard that the matrix multiplication is defined only when the size of matrix is equal. The size matters because composition is defined only when the domain and target are equal.

Definition 4.5 (addition and s-multiplication)**.** Recall that $\mathrm{Hom}(V, W)$ is also a vector space, that is, the addition and s-multiplication is important. They are represented as addition and s-multiplication in the matrix: For $A, B \in \mathrm{Hom}(V, W)$,

$$A +_{\hom(V,V)} V \longrightarrow \mathbf{A} + \mathbf{B} = \begin{pmatrix} A^1{}_1 + B^1{}_1 & A^1{}_2 + B^1{}_2 & \dots & A^1{}_n + B^1{}_n \\ A^2{}_1 + A^2{}_1 & A^2{}_2 + B^2{}_2 & \dots & A^2{}_n + B^2{}_n \\ \vdots & \vdots & \ddots & \vdots \\ A^m{}_1 + B^m{}_1 & A^m{}_2 + B^m{}_2 & \dots & A^m{}_n + B^m{}_n \end{pmatrix}, \tag{4.16}$$

and for $c \in \mathbb{K}$

$$c \heartsuit_{\mathrm{Hom}(V,W)} A \longrightarrow c\mathbf{A} = \begin{pmatrix} cA^1{}_1 & cA^1{}_2 & \dots & cA^1{}_n \\ cA^2{}_1 & cA^2{}_2 & \dots & cA^2{}_n \\ \vdots & \vdots & \ddots & \vdots \\ cA^m{}_1 & cA^m{}_2 & \dots & cA^m{}_n \end{pmatrix} \tag{4.17}$$

One-form is the special case of linear map whose target space is the scalar. Since $\dim \mathbb{K} = 1$, the one-form in $V^\vee$ should be represented with $1 \times m$ matrix:

$$\mathbf{A} = \Big(A^1{}_1 \; A^1{}_2 \; \dots \; A^1{}_n \Big). \tag{4.18}$$

Let us specify the basis of $\mathbb{K}$ as f_1:

$$A(v) = \Big(\sum_i A^1{}_i v^i\Big) f_1. \tag{4.19}$$

Remark. The basis f_1 can be chosen any nonzero number. Note that

$$f_1 = 1 \iff A(v) = \sum_i A^1{}_i v^i. \tag{4.20}$$

For convenience, everyone choose $f_1 = 1$ even without conscience. On the other hand, we also know that the dual space is also a vector space. Since $\dim V = \dim V^\vee$,[6] we also need n numbers to express it. We lazily call it also vector.

Definition 4.6 (row vector)**.** Let $\{\alpha^1, \dots, \alpha^n\}$ be the dual basis one-form for $\{e_1, \dots, e_n\}$. The one-form $\omega = \omega_1\alpha^1 + \dots + \omega_n\alpha^n \in V^\vee$ is expressed

$$\boldsymbol{\omega} = \begin{pmatrix}\omega_1 & \omega_2 & \dots & \omega_n\end{pmatrix}. \tag{4.21}$$

It is called *row vector*. Moreover, the matrix multiplication now expresses $f_1 = 1$ case.

Proof. Let us express it in the linear combination

$$\begin{aligned}\omega(v) &= \sum_i \omega_i \alpha^i\Big(\sum_j v^j e_j\Big)\\ &= \sum_{i,j} \omega_i v^j \alpha^i(e_j)\\ &= \sum_{i,j} \omega_i v^j \delta^i_j\\ &= \sum_i \omega_i v^i.\end{aligned}$$

where in the second line, linearity is used. □

Box 4.1: Notations for one-form acting on vector

For $v \in V$, $\omega \in V^\vee$, we have following equivalent expressions for $\omega(v) \in \mathbb{K}$:

$$\omega(v) \iff \begin{pmatrix}\omega_1 & \omega_2 & \dots & \omega_n\end{pmatrix}\begin{pmatrix}v^1\\ v^2\\ \vdots\\ v^n\end{pmatrix} = \sum_i \omega_i v^i \tag{4.22}$$

The first is notation we have used, the second is Dirac notation and the third is matrix notation.

Remark. Note that is not the inner product. In fact, we do not introduce the inner product yet. Again, we cannot say anything about the angle between two vectors.

[6]We only consider finite dimensional case, when I do not specify dimensions.

4.2 Expression in the Inner Product Space

From now, we assume V, W are inner product $\mathbb{K}$-vector space. Now I introduce the most famous notation in quantum mechanics, the Dirac notation. It is useful notation in the inner product space [7]. Dirac notation uses *ket* notation:

$$|v\rangle := v \in V \tag{4.23}$$

It is called *ket vector*. For linear map, we skip the bracket ( ) when they act. That is for $A \in \mathrm{Hom}(V, W)$:

$$|Av\rangle := A(v) \in W \tag{4.24}$$

The usefulness appears when we write dual basis. Let α^i be the dual basis for the orthonormal basis e_i. Recall that the slot notation,

$$\langle e_i, \cdot\,\rangle :\Leftrightarrow\ \alpha^i. \tag{4.25}$$

We do not need to introduce new alphabet if we have expression that make us distinguish one-form and vector. That is,

$$\langle e_i| := \langle e_i, \cdot\,\rangle \in V^\vee \tag{4.26}$$

we call this notation *bra*. Also, the slot notation allows us to introduce dual vector corresponding to v

$$\langle v| :\Leftrightarrow \langle v, \cdot\,\rangle. \tag{4.27}$$

It is called *bra vector*. For $v, w \in V$, the inner product is denoted:

$$\langle v|w\rangle \ :\Leftrightarrow\ \langle v, w\rangle. \tag{4.28}$$

It is combination of bar and cat called *bracket*.

4.2.1 Real Vector Space

For the real space, inner product is symmetric bilinear map. Therefore,

$$v_j = \langle v|e_j\rangle = \left\langle \sum_i v^i e_i \middle| e_j \right\rangle = \sum_{i,j} v^j \langle e_i|e_j\rangle = \sum_{i,j} \delta_{ij} v^j \tag{4.29}$$

where linearity in the first slot and orthonormality is used. Therefore, we identify one form in matrix form

$$\langle v| = \begin{pmatrix} v^1 & v^2 & \dots & v^n \end{pmatrix}. \tag{4.30}$$

Note that it is not only just a map, it is isomorphism.

[7] More precisely, this is useful when there is an bijection between vector and one-form (♯ and ♭ operator).

Box 4.2: Transpose of a vector

In real vector space, the map from vector to 'corresponding' one-form in real vector space is called the *transpose*:

$$\mathsf{T} : V \to V^\vee \quad \text{by } |v\rangle \mapsto |v\rangle^\mathsf{T} := \langle v| \tag{4.31}$$

Recall that $V^{\vee\vee} \cong_{\text{vec}} V$. Therefore, we can use the *transpose* also for map from $V^\vee$ to V.

$$\mathsf{T} : V^\vee \to V \quad \text{by } \langle v| \mapsto \langle v|^\mathsf{T} := |v\rangle. \tag{4.32}$$

The transpose map is an isomorphism whose inverse is itself.

Remark. In matrix notation, we just change column vector to row vector and vice versa:

$$\begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}^\mathsf{T} = \begin{pmatrix} v^1 & v^2 & \dots & v^n \end{pmatrix}, \quad \begin{pmatrix} v^1 & v^2 & \dots & v^n \end{pmatrix}^\mathsf{T} = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} \tag{4.33}$$

Remark. Let us recall that the inner product is symmetric. The inner product is

$$\langle v|w\rangle = \langle w|v\rangle \tag{4.34}$$

On the other hand, the r.h.s. can also be the inner product of $|w\rangle^\mathsf{T}$ and $\langle v|^\mathsf{T}$.

Now let us generalise this to the linear map.

Box 4.3: Transpose of Linear Map

Let $A \in \text{Hom}(V, W)$. We define the *transpose* map $\mathsf{T} : \text{Hom}(V, W) \to \text{Hom}(W^\vee, V^\vee)$ such that

$$\forall\, |v\rangle \in V : \forall\, |w\rangle \in W : \langle A^\mathsf{T} w|v\rangle := \langle w|Av\rangle. \tag{4.35}$$

Since the inner product is symmetric, this implies

$$\forall\, |v\rangle \in V : \forall\, |w\rangle \in W : \langle v|A^\mathsf{T} w\rangle = \langle Aw|v\rangle. \tag{4.36}$$

That is $\mathsf{T} : \text{Hom}(W^\vee, V^\vee) \to \text{Hom}(V, W)$ also defined automatically. Again, transpose is isomorphism whose inverse is itself.

Remark. In the matrix notation, we just change the row and column:

$$\mathbf{A}^\mathsf{T} = \begin{pmatrix} A^1{}_1 & A^1{}_2 & \dots & A^1{}_n \\ A^2{}_1 & A^2{}_2 & \dots & A^2{}_n \\ \vdots & \vdots & \ddots & \vdots \\ A^m{}_1 & A^m{}_2 & \dots & A^m{}_n \end{pmatrix}^\mathsf{T} = \begin{pmatrix} A^1{}_1 & A^2{}_1 & \dots & A^m{}_1 \\ A^1{}_2 & A^2{}_2 & \dots & A^m{}_2 \\ \vdots & \vdots & \ddots & \vdots \\ A^1{}_n & A^2{}_n & \dots & A^m{}_n \end{pmatrix}. \tag{4.37}$$

4.2.2 Complex Vector Space

For the complex vector space, inner product is symmetric sesquilinear map. Therefore,

$$v_j = \langle v|e_j\rangle = \left\langle \sum_i v^i e_i \middle| e_j \right\rangle = \sum_{i,j} \overline{v^j}\langle e_i|e_j\rangle = \sum_{i,j} \delta_{ij}\overline{v^j} \tag{4.38}$$

where linearity in the first slot and orthonormality is used. Therefore, we identify one form in matrix form

$$\langle v| = \begin{pmatrix} \overline{v^1} & \overline{v^2} & \dots & \overline{v^n} \end{pmatrix}. \tag{4.39}$$

Note that it is not only just a map, it is isomorphism.

Box 4.4: Hermitian Adjoint of a vector

In complex vector space, the map from vector to 'corresponding' one-form in real vector space is called the *Hermitian adjoint* or just *Hermitian* of vector:

$$\dagger : V \to V^\vee \quad \text{by } |v\rangle \mapsto |v\rangle^\dagger := \langle v| \tag{4.40}$$

Recall that $V^{\vee\vee} \cong_{\text{vec}} V$. Therefore, we can use the *Hermitian conjugate* also for map from $V^\vee$ to V.

$$\dagger : V^\vee \to V \quad \text{by } \langle v| \mapsto \langle v|^\dagger := |v\rangle. \tag{4.41}$$

The Hermitian adjoint map is an isomorphism whose inverse is itself.

Remark. In matrix notation, we just change column vector to row vector and do the complex conjugate to each element, and vice versa:

$$\begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}^\dagger = \begin{pmatrix} \overline{v^1} & \overline{v^2} & \dots & \overline{v^n} \end{pmatrix}, \quad \begin{pmatrix} v^1 & v^2 & \dots & v^n \end{pmatrix}^\dagger = \begin{pmatrix} \overline{v^1} \\ \overline{v^2} \\ \vdots \\ \overline{v^n} \end{pmatrix}. \tag{4.42}$$

In matrix we usually say *Hermitian conjugate* instead of Hermitian adjoint.

Remark. Let us recall that the inner product is symmetric. The inner product is

$$\langle v|w\rangle = \overline{\langle w|v\rangle} \tag{4.43}$$

On the other hand, the r.h.s. can also be the inner product of $|w\rangle^\dagger$ and $\langle v|^\dagger$.

Now let us generalise this to the linear map.

Box 4.5: Hermitian Adjoint of Linear Map

Let $A \in \mathrm{Hom}(V, W)$. We define the *Hermitian Adjoint* map $\dagger : \mathrm{Hom}(V, W) \to \mathrm{Hom}(W^\vee, V^\vee)$ such that

$$\forall\, |v\rangle \in V : \forall\, |w\rangle \in W : \langle A^\dagger w|v\rangle \coloneqq \langle w|Av\rangle. \tag{4.44}$$

Since the inner product is symmetric, this implies

$$\forall\, |v\rangle \in V : \forall\, |w\rangle \in W : \langle v|A^\dagger w\rangle = \langle Aw|v\rangle. \tag{4.45}$$

That is $\dagger : \mathrm{Hom}(W^\vee, V^\vee) \to \mathrm{Hom}(V, W)$ also defined automatically. Again, Hermitian adjoint is isomorphism whose inverse is itself.

Remark. In the matrix notation, we just change the row and column and take complex conjugate for each elements:

$$\mathbf{A}^\dagger = \begin{pmatrix} A^1{}_1 & A^1{}_2 & \dots & A^1{}_n \\ A^2{}_1 & A^2{}_2 & \dots & A^2{}_n \\ \vdots & \vdots & \ddots & \vdots \\ A^m{}_1 & A^m{}_2 & \dots & A^m{}_n \end{pmatrix}^\dagger = \begin{pmatrix} \overline{A^1{}_1} & \overline{A^2{}_1} & \dots & \overline{A^m{}_1} \\ \overline{A^1{}_2} & \overline{A^2{}_2} & \dots & \overline{A^m{}_2} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{A^1{}_n} & \overline{A^2{}_n} & \dots & \overline{A^m{}_n} \end{pmatrix}. \tag{4.46}$$

Remark. We have learnt that Hermitian conjugate is transpose and complex conjugate:

$$\mathbf{A}^\dagger = \overline{\mathbf{A}}^\intercal \tag{4.47}$$

Remark. One bad thing of the matrix representation is that it can make misconception that the relation between Hermitian conjugate and transpose. You must keep in mind that Hermitian conjugate and transpose are conceptually the same operation acting on vector spaces with different scalars (complex and real).

Remark. More problem is that in the matrix representation, it is not possible to distiguish the linear map if the dimension of the target space is the same. Particularly, in physics, $\mathrm{Hom}(V, V^\vee)$ and $\mathrm{End}(V)$ appears a lot. If you use only the matrix representation, there is no way to distinguish them.

4.3 Abstract Index Notation

In general relativity, we are interested in the geometric quantities. The geometric quantities are vectors (including the (multi-)linear maps) of one type of vector space. Therefore, one needs the easy notation which distinguish the vector and one-form and linear maps without ambiguity. The famous notation is the abstract index notation. We write vector with one upper index, one-form with lower index and multilinear maps with multiple indices:[8]

$$\begin{aligned} v^a &\quad \text{means vector } v; \\ \omega_a &\quad \text{means one-form } \omega; \\ g_{ab} &\quad \text{means bilinear form } g; \\ \ldots & \end{aligned}$$

One can also write the image of linear map by using the same indices. For example,

$$\begin{aligned} \omega_a v^a &\ :\Leftrightarrow\ \omega(v) \in \mathbb{K} \\ g_{ab} v^a &\ :\Leftrightarrow\ g(v, \cdot\,) \in V^\vee \\ g_{ab} v^a w^b &\ :\Leftrightarrow\ g(v, w) \in \mathbb{K} \\ \ldots & \end{aligned}$$

And this leads to the Einstein summation convention

Box 4.6: Einstein Summation Convention

If you see one upper one lower indices with same alphabet in the components, summation is abbreviated. For example, let $\omega_a = \omega_1 \alpha^1 + \ldots + \omega_n \alpha^n \in V^\vee$ and $v^a = v^1 e_1 + \ldots + v^n e_n$. Then,

$$\omega_i v^i :\Leftrightarrow \sum_i \omega_i v^i = \omega_1 v^1 + \ldots + \omega_n v^n. \tag{4.48}$$

where each $\omega_i \in \mathbb{K}$ and $v^i \in \mathbb{K}$ are components.

Box 4.7: Musical Map

There is a map called the *musical map* or *musical isomorphism* which relates vector to one-form.

$$\sharp : V^\vee \to V \qquad \text{by } \omega_a \mapsto \omega_a^\sharp := \omega^a. \tag{4.49}$$

$$\flat : V \to V^\vee \qquad \text{by } v^a \mapsto (v^a)^\flat = v_a. \tag{4.50}$$

The meaning is clear in the abstract index notation because each raise/lower the indices.

[8] Not to confuse with the components. Components are just scalars.

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