Spectral Decomposition II

Sang-Shin Baak

Korea University, Natural Science Research Institute,
Sejong 30019, Korea

E-mail: psn5210@naver.com

Abstract: References are of course my favorite [1, 2] and additionally, [3–5]. I find [6] is very good. In this note, we will learn the examples in spectral decompositions. The first is normal operators in complex vector space. It is always diagonal and it is the one we meet in the most cases of quantum mechanics textbook. The other is a linear map in complex vector space.

Contents

We will learn specific example of spectral decomposition. In fact, spectral decomposition is easier in the complex vector space.

1 Spectral Decomposition of Complex Vector Space

We will see some example of diagonalisation here. Not all operators are diagonalisable. Therefore, we will first tell about the operator satisfying special property, which will be shown diagonalisable later.

Box 1.1: Normal Operator

$A \in \mathrm{End}(V)$ is called *normal* when it commutes with its adjoint. That is, if it satisfies

$$0 = [A, A^\dagger] := AA^\dagger - A^\dagger A. \tag{1.1}$$

Theorem 1.1. Let $A \in \mathrm{End}(V)$. Then,

$$\big(\forall v \in V : \quad \|Av\| = \|A^\dagger v\|\big) \iff A \text{ is normal.} \tag{1.2}$$

Proof.

$$\|Av\|^2 = \langle Av|Av\rangle = \langle v|A^\dagger Av\rangle \tag{1.3}$$

$$\|A^\dagger v\|^2 = \langle A^\dagger v|A^\dagger v\rangle = \langle v|AA^\dagger v\rangle \tag{1.4}$$

($\Longrightarrow$) is assume l.h.s. of above equation is the same and show r.h.s is the same. ($\Longleftarrow$) is just opposite direction. □

Theorem 1.2. Let $A \in \mathrm{End}(V)$ be a normal operator and let $U \subseteq V$ be an invariant subspace of A. Then U reduces A.

Proof. This can be proved more easily by using matrix representation. Since U is an invaraiant subspace of A, we have a basis such that

$$\forall |u\rangle \in U : |u\rangle = \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} \qquad \text{which yields} \qquad A = \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}. \tag{1.5}$$

Then

$$|Au\rangle = \begin{pmatrix} \mathbf{A}_{11}\mathbf{u} \\ 0 \end{pmatrix}. \tag{1.6}$$

On the other hand, by using the matrix representation of adjoint, we can also show that

$$|A^\dagger u\rangle = \begin{pmatrix} \mathbf{A}_{11}^\dagger\mathbf{u} \\ \mathbf{A}_{12}^\dagger\mathbf{u} \end{pmatrix}. \tag{1.7}$$

Since A is normal, $\|Av\| = \|A^\dagger v\|$. Using inner product in the matrix representation, (matrix multiplication with complex conjugate transpose)

$$\|Av\|^2 = \|\mathbf{A}_{11}\mathbf{u}\|_{\mathrm{mat}}^2 \tag{1.8}$$

$$= \|A^\dagger v\|^2 = \|\mathbf{A}_{11}^\dagger\mathbf{u}\|_{\mathrm{mat}}^2 + \|\mathbf{A}_{12}^\dagger\mathbf{u}\|_{\mathrm{mat}}^2 \tag{1.9}$$

On the other hand,

$$\|\mathbf{A}_{11}\mathbf{u}\|_{\text{mat}}^2 = \|\mathbf{A}_{11}^\dagger\mathbf{u}\|_{\text{mat}}^2 \tag{1.10}$$

Therefore,

$$\forall|u\rangle \in U : \|\mathbf{A}_{12}^\dagger\mathbf{u}\|_{\text{mat}}^2 = 0 \implies \mathbf{A}_{12} = 0. \tag{1.11}$$

Then, $\mathbf{A} = \mathbf{A}_{11} \oplus \mathbf{A}_{22}$ is block diagonal form. Therefore we conclude that U reduces A. □

Theorem 1.3. Let $A \in \text{End}(V)$ be normal. Then,

$$A|v\rangle = \lambda|v\rangle \iff A^\dagger|v\rangle = \overline{\lambda}|v\rangle \tag{1.12}$$

Proof. If A is normal, $A - \lambda\mathbb{1}$ is also normal since

$$\begin{aligned}
(A - \lambda\mathbb{1})(A - \lambda\mathbb{1})^\dagger &= (A - \lambda\mathbb{1})(A^\dagger - \overline{\lambda}\mathbb{1}) \\
&= AA^\dagger - \lambda A^\dagger - \overline{\lambda}A + |\lambda|^2 \\
&= A^\dagger A - \overline{\lambda}A - \lambda A^\dagger + |\lambda|^2 \\
&= (A^\dagger - \overline{\lambda}\mathbb{1})(A - \lambda\mathbb{1}) \\
&= (A - \lambda\mathbb{1})^\dagger(A - \lambda\mathbb{1})
\end{aligned}$$

Then we have

$$\forall v \in V : \quad \|(A - \lambda\mathbb{1})v\| = \|(A - \lambda\mathbb{1})^\dagger v\| = \|(A^\dagger - \overline{\lambda}\mathbb{1})v\| \tag{1.13}$$

If $|v\rangle$ is an eigenvector of A corresponding to the eigenvalue λ,

$$0 = \|(A - \lambda\mathbb{1})v\| = \|(A^\dagger - \overline{\lambda}\mathbb{1})v\| \tag{1.14}$$

Therefore, $\overline{\lambda}$ is an eigenvalue of $A^\dagger$. □

Theorem 1.4. An eigenspace of normal operators *reduces* that operator. Moreover, they are mutually orthogonal.

Proof. First, let us see eigenspace reduces that operator. Let $A \in \text{End}(V)$ be that normal operator and E_a be the eigenspace corresponding to eigenvalue a. Since s-multiplication is closed operation,

$$\forall|v\rangle \in E_a : |Av\rangle = a|v\rangle \in E_a. \tag{1.15}$$

We have shown that the invariant subspace of normal operator reduces that operator.Since eigenspace is invariant subspace, for normal operator, it reduces A.
Next, we want to check if it is mutually orthogonal. Let $a \neq b$.

$$\begin{aligned}
\forall\, v_a \in E_a : \forall\, v_b \in E_b : \langle v_a|Av_b\rangle &= \langle v_a|bv_b\rangle = b\langle v_a|v_b\rangle \\
&= \langle A^\dagger v_a|v_b\rangle = \langle \overline{a}v_a|v_b\rangle = a\langle v_a|v_b\rangle
\end{aligned}$$

Therefore, using $a \neq b$,

$$\forall\, v_a \in E_a : \forall\, v_b \in E_b : \langle v_a|v_b\rangle = 0 \tag{1.16}$$

We conclude that $E_a \perp E_b$. □

Box 1.2: Complex Spectral Decomposition

Let $A \in \mathrm{End}(V)$ be a normal operator. And let $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues, and $E_1 \ldots, E_r$ be the corresponding eigenspaces. Then,

$$V = \bigoplus_{i=1}^{r} E_i \tag{1.17}$$

Moreover, if P_j is the projection operator on V onto E_j, $\{P_i\}_{i=1}^r$ is a complete set of orthogonal projection operators. Particularly, we have

$$A = \sum_{j=1}^{r} \lambda_j P_j \tag{1.18}$$

Proof. Others are already shown in previous theorems. The only thing we need to check is the completeness. Let $E = E_1 \oplus \ldots \oplus E_r$. Then, M reduces A. Since M is normal, $M^\perp$ also reduces A. If $M^\perp$ is not trivial, (not zero dimension), one can write the characteristic equation for operator A acting on the $M_\perp$ (restriction of A). If it is not trivial vector space, it must have at least one eigenvalue and eigenvector. But, $M^\perp$ is defined as a space which is orthogonal to all eigevectors. Therefore, $M^\perp$ is trivial ($M^\perp = \{|0\rangle\}$. □

Theorem 1.5. Let $(V, \langle\ ,\ \rangle)$ be an inner product space and $A \in \mathrm{End}(V)$. Then,

$$A \text{ is diagonalisable.} \qquad \Longleftrightarrow \qquad A \text{ is normal.} \tag{1.19}$$

Proof. For each subspace E_j, take orthonormal basis $\{|e_j^s\rangle\}$. Then,

$$\begin{aligned}
\langle e_s^j | A e_{j'}^{s'} \rangle &= \langle e_s^j | \sum_k \lambda_k P_k e_{j'}^{s'} \rangle \\
&= \sum_k \lambda_k \langle e_s^j | P_k e_{j'}^{s'} \rangle \\
&= \sum_k \lambda_k \delta_{j',k} \langle e_s^j | e_{j'}^{s'} \rangle && (1.20) \\
&= \lambda_k \langle e_s^j | e_{j'}^{s'} \rangle && (1.21) \\
&= \lambda_k \delta_{j'}^{j} \delta_{s'}^{s}. && (1.22)
\end{aligned}$$

Therefore, A is diagonal in the orthonormal basis of each eigenspace.

Conversely, if A is diagonalisable, there is a basis which makes the matrix representation A diagonal. In that basis $A^\dagger$ is also diagonal matrix, and therefore, the matrix multiplication commutes. □

Theorem 1.6. There are two important cases of normal operators in the complex inner product space.

- *Hermitian* operators ($H^\dagger = H$) have real eigenvalues ($\lambda \in \mathbb{R}$);
- *Unitary* operators ($U^\dagger U = \mathbb{1}$) have eigenvalues whose magnitude is equal to 1. ($|\lambda|^2 = 1$)

Proof. Let $H \in \mathrm{End}(V)$ be an Hermitian operator. Let $|v\rangle \in V$ be a eigenvector corresponding to the eigenvalue λ. Then,

$$\lambda\langle v|v\rangle = \langle v|\lambda v\rangle = \langle v|Hv\rangle = \langle H^\dagger v|v\rangle = \langle Hv|v\rangle = \langle \lambda v|v\rangle = \overline{\lambda}\langle v|v\rangle. \tag{1.23}$$

Since eigenvector $|v\rangle$ is nonzero, $\lambda - \overline{\lambda} = 0$ which means $\lambda \in \mathbb{R}$.

Let U be a unitary operator. Let $|v\rangle \in V$ be a eigenvector corresponding to the eigenvalue λ. Then,

$$\begin{aligned} \|Uv\|^2 &= \langle Uv|Uv\rangle = \langle \lambda v|\lambda v\rangle = |\lambda|^2\langle v|v\rangle \\ &= \langle v|U^\dagger Uv\rangle = \langle v|\mathbb{1}v\rangle = \langle v|v\rangle. \end{aligned}$$

Again, $|v\rangle$ is nonzero. Therefore, $|\lambda|^2 = 1$. □

Lemma 1.7. An Hermitian operator is *positive* iff all its eigenvalues are positive.

Lemma 1.8. An Hermitian matrix can be brought to diagonal form by means of a unitary transform.

Example 1.8.1 (Charged Particle in a Magnetic Field)**.** When we were freshman, we have learnt that the Lorentz force on a charged particle with charge q by magnetic field $\mathbf{B}$ is $F = q\mathbf{v}\times\mathbf{B}$. For simplicity, we assume charged particle moves in xy-plane, and the magnetic field is directing z direction, i.e., $\mathbf{B} = B\hat{z}$. In the 'usual' basis, that is the basis vectors align with each axis, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{qB}{m}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -i\omega\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{1.24}$$

where $\omega = qB/m$ and m is the mass of the particle. Note that this is classical mechanics problem, but we have an Hermitian matrix. Let us multiply i on both side. Then, we get

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & i\omega \\ -i\omega & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} := H\begin{pmatrix} v_x \\ v_y \end{pmatrix}. \tag{1.25}$$

Now, let us diagonalise H. First, we solve characteristic equation $\det(H - \lambda\mathbb{1}) = 0$ to obtain eigenvalues. For $\lambda = \omega$ we solve

$$0 = H - \lambda\mathbb{1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\omega & i\omega \\ -i\omega & \omega \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \implies -a + ib = 0. \tag{1.26}$$

Therefore, we have eigenvector $|e_1\rangle = \frac{1}{\sqrt{2}}(1, -i)^\intercal$. Similarly, we get $|e_2\rangle = \frac{1}{\sqrt{2}}(1, i)^\intercal$. The time evolution of each components are given,

$$v_1(t)|e_1\rangle = v_{1,0}\mathrm{e}^{-i\omega t}|e_1\rangle, \qquad v_2(t)|e_2\rangle = v_{2,0}\mathrm{e}^{i\omega t}|e_2\rangle. \tag{1.27}$$

where $|v\rangle = v_1|e_1\rangle + v_2|e_2\rangle$ and $v_{1,0} = v_1(t=0), v_{2,0} = v_2(t=0)$. In fact, the diagonalisation procedure is just representing this by using basis change. We want to work in the new basis whose basis is these eigenvectors. Therefore, we use the transformation law for the components. The transformation matrix is

$$U = \begin{pmatrix} \langle e_1|e_x\rangle & \langle e_1|e_y\rangle \\ \langle e_2|e_x\rangle & \langle e_2|e_y\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{1.28}$$

Recall that $U^{-1} = U^\dagger$. The equation of motion (1.25) can be written as

$$U^\dagger i\frac{\mathrm{d}}{\mathrm{d}t}U\begin{pmatrix} v_x \\ v_y \end{pmatrix} = U^\dagger U H U^\dagger U\begin{pmatrix} v_x \\ v_y \end{pmatrix} = U^\dagger\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{1.29}$$

where we used the fact that U is independent of t. The solution of this equation can easily obtained

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = U^\dagger\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = U^\dagger\begin{pmatrix} v_{1,0}\mathrm{e}^{-i\omega t} \\ v_{2,0}\mathrm{e}^{i\omega t} \end{pmatrix} \tag{1.30}$$

In general, we usually meet several operators, not just one. If there is a basis making operators we are interested in the diagonal form, life would be easy.

Definition 1.1 (simultaneouslu diagonalisable)**.** If two operators $A, B \in \mathrm{End}(V)$ can be written in terms of the same set of projection operators,

$$A = \sum_{i=1}^{n} A_i P_i, \qquad B = \sum_{i=1}^{n} B_i P_i \tag{1.31}$$

where $\{P_i\}_{i=1}^n$ is the complete orthonormal set of projection operators. Then we say A and B are *simultaneously diagonalisable*.

Box 1.3: Simultaneously Diagonalisable

Let both $A, B \in \mathrm{End}(V)$ are normal operators. Then they are simultaneously diagonalisable iff they are commute, i.e., $[A, B] = 0$.

Proof. ($\Longrightarrow$) We then have special basis (set of orthonormal bases in each eigenspace) which makes both A, B diagonal. The diagonal matrix commutes.
($\Longleftarrow$) Since every normal operators in complex inner product vector space is diagonalisable, we can write

$$A = \sum_{j=1}^{r} \alpha_j P_j, \qquad B = \sum_{i=1}^{s} \beta_i Q_i \tag{1.32}$$

where α_j, β_i are eigenvalues, and $\{P_i\}_{i=1}^r$ and $\{Q_i\}_{i=1}^s$ are the complete set of projection operators on V to each corresponding eigenspace. We define operator $R_{ji} = P_j Q_i$ which sends a vector in eigenspace of B corresponding to β_i to vector in eigenspace of A corresponding to α_i. We want to show that this is projection operator to the common eigenspace $E_{\alpha_j} \cap E_{\beta_i}$ using the condition. From the condition,

$$\forall i \in \{1, \ldots, s\} : \forall\, |u\rangle \in E_{\beta_i} : |BAu\rangle = |ABu\rangle = |A\beta_j u\rangle = \beta_j |Au\rangle. \tag{1.33}$$

Therefore, E_{β_j} is also an invariant subspace of A. In fact, it reduces A and

$$\forall i \in \{1, \ldots, s\} : \forall |v\rangle \in V : |AQ_i u\rangle = |Q_i A u\rangle. \tag{1.34}$$

Therefore, we get $[A, Q_i] = 0$. Similarly, we can also obtain $[P_j, Q_i] = 0$. Using this, we can show that R_{ji} is a projection operator (Hermitian idempotent)

$$R_{ji}^\dagger = Q_i^\dagger P_j^\dagger = Q_i P_j = P_j Q_i = R_{ji} \tag{1.35}$$

$$(R_{ji})^2 = P_j Q_i P_j Q_i = (P_j)^2 (Q_i)^2 = P_j Q_i = R_{ji}. \tag{1.36}$$

It is easy to show that

$$\sum_{j=1}^{r} R_{ji} = Q_i, \qquad \sum_{i=1}^{s} R_{ji} = P_j, \qquad \sum_{j,i} R_{ji} = \mathbb{1}. \tag{1.37}$$

Therefore, we have

$$V = \bigoplus_{j,i} E_{\alpha_j} \cap E_{\beta_i}, \tag{1.38}$$

and

$$A = \sum_j \alpha_j P_j = \sum_{j,i} \alpha_j R_{ji}, \qquad B = \sum_i \beta_i Q_i = \sum_{j,i} \beta_i R_{ji}. \tag{1.39}$$

□

2 Linear Operator in Complex Vector Space

Normal operators are quite special. We sometimes meet operators which are not normal. For example, the creation/annihilation operator in quantum mechanics is not normal. In this section, let us see what we can tell about more general case.

2.1 Minimal Polynomial

Before telling about the complex vector space, let us see some general arguments. Recall that we represent the composition of map as a matrix multiplication. Let $A \in \mathrm{End}(V)$. Then we write

$$A^n :\Leftrightarrow \underbrace{A \circ \ldots \circ A}_{n \text{ times}}, \qquad A^0 = \mathbb{1} \tag{2.1}$$

Definition 2.1 (polynomial of an operator)**.** The *polynomial* $p(A)$ of an operator A is

$$p(A) = a_m A^m + a_{m-1} A^{m-1} + \ldots + a_0 \mathbb{1}, \tag{2.2}$$

where $a_m \neq 0$. The maximum nonzero power of A, m is called a *degree* of $p(A)$ and write $\deg p$.

Let $\dim V = n < \infty$. Recall that $\dim \mathrm{End}(V) = n^2$. Therefore, we can have maximally n^2 independent vectors[1]. Then $\{A^0 = \mathbb{1}, A^1, \ldots, A^{n^2}\}$ is the set of linearly dependent vectors. In other words,

$$\exists a_0, \ldots, a_{n-1} : A^{n^2} = \sum_{j=0}^{\infty} b_j A^j \tag{2.3}$$

[1]Recall that linear maps also form a vector space. To emphasise it, I write vector instead of linear map.

Definition 2.2 (annihilating polynomial)**.** The *annihilating polynomial* is a polynomial of the operators whose value is zero.

We always have the annihilating polynomial

$$p(A) = \sum_{i=0}^{n^2} a_i A^i = 0. \tag{2.4}$$

But n^2 looks quite big. There can be a smaller degree of annihilating polynomial.

Box 2.1: Cayley-Hamilton Theorem

Let $A \in \mathrm{End}(V)$ and $\dim V = n$. Let us write characteristic polynomial

$$\chi_A(\lambda) \coloneqq \det(\lambda \mathbb{1} - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_0/ \tag{2.5}$$

Then

$$\chi_A(A) = A^n + c_{n-1}A^{n-1} + \ldots + c_0\mathbb{1} = 0. \tag{2.6}$$

In other words, characteristic polynomial is the annihilating polynomial.

Proof. Since we characteristc polynomial is defined with determinant, we will work with the matrix representation. If A is diagonalisable, we can simply diagonalise it using unitary transformation. Let $A = UAU^\dagger$ with $D = \mathrm{diag}[\lambda_1, \ldots \lambda_r]$. Then $A^n = UD^nU^\dagger$

$$\begin{aligned} \chi_A(A) &= U(D^n + c_{n-1}D^{n-1} + \ldots + c_0\mathbb{1})U^\dagger \\ & U\mathrm{diag}[\chi_A(\lambda_1), \ldots, \chi_A(\lambda_r)]U^\dagger \\ &= 0. \end{aligned}$$

If it is nondiagonalisable, it is more complicate. Recall that the inverse matrix can be expressed with adjugate matrix

$$(\lambda\mathbb{1} - A)^{-1} = \frac{1}{\chi_A(A)} B \coloneqq \frac{1}{\det(\lambda\mathbb{1} - A)} \mathrm{adj}(\lambda\mathbb{1} - A). \tag{2.7}$$

Since adjugate matrix is the transpose of the cofactor matrix. Then, we can expand it with order of λ:

$$B = \sum_{i=0}^{n-1} \lambda^i B_i, \qquad \text{where } B_i \in \mathrm{Mat}_{n-1}(\mathbb{C}). \tag{2.8}$$

Then, we have

$$\chi_A(\lambda)\mathbb{1} = (\lambda\mathbb{1} - A)B = (\lambda\mathbb{1} - A)\sum_{i=0}^{n-1}\lambda^i B_i = \lambda^n B_{n-1} + \sum_{n=1}^{n-1}\lambda^i(B_{i-1} - AB_i) - AB_0. \tag{2.9}$$

Comparing coefficients of λ gives

$$\mathbb{1} = B_{n-1}, \qquad \text{for } i \in \{1, \ldots, n-1\}\text{: } c_i\mathbb{1} = B_{i-1} - AB_i, \qquad c_0\mathbb{1} = -AB_0 \tag{2.10}$$

Therefore

$$\begin{aligned}\chi_A(A) &= A^n + c_{n-1}A^{n-1} + \ldots + c_0\mathbb{1} \\ &= A^n B_{n-1} + \sum_{i=1}^{n-1}(B_{i-1} - AB_i)A^i - AB_0 \\ &= 0.\end{aligned}$$

□

Sincs n^2 is finite, there is a minimum degree annihilating polynomial

$$p(A) = \sum_{i=0}^{m} a_i A^i = 0, \qquad m \leq 0. \tag{2.11}$$

Note that it is unique up to multiplication of constants.

Box 2.2: Minimal Polynomial

Let $A \in \mathrm{End}(V)$. The *minimal polynomial* of A is a minimal degree, monic (coefficient of highest power is 1), annihilating polynomial. That is,

$$p(A) = A^m + a_{m-1}A^{m-1} + \ldots + a_0 = 0. \tag{2.12}$$

Theorem 2.1 (Factoring Annihilating Polynomial)**.** We can factor annihilaing polynomial with minimal polynomial.

Proof. Let $s(A)$ be an annihilating polynomial of $A \in \mathrm{End}(V)$ and $p(A)$ be a minimal polynomial. Since $\deg p \leq \deg s$, we can write it as

$$s = pq + r. \tag{2.13}$$

where the $\deg r < \deg p$.

$$0 = s(A) = 0q(A) + r(A) = r(A). \tag{2.14}$$

Therefore, we can conclude $s(A) = p(A)q(A)$. □

Lemma 2.2. A minimal polynomial is unique.

Proof. Let $p_1(A)$ and $p_2(A)$ be two distinct minimal polynomial. Then p_1 factors p_2. But they have the same degree so they p_2 is just scalar multiple of p_1. Since minimal polynomial is monic, the only possible scalar is 1. Therefore, $p_1 = p_2$. □

There is a special case that the minimal polynomial is just mth power of the matrix.

Definition 2.3 (Nilpotent)**.** An operator $N \in \mathrm{End}(V)$ is called *nilpotent* if

$$\exists m \in \mathbb{Z}_+ : \qquad N^m = 0. \tag{2.15}$$

Example 2.2.1. For finite-dimensional vector space, annihilation operator a is a nilpotent operator.

Theorem 2.3. Let $A \in \mathrm{End}(V)$. Then

$$\ker A \subseteq \ker A^2 \subseteq \dots \tag{2.16}$$

Lemma 2.4. For nilpotent operator $N \in \mathrm{End}(V)$ whose degree of minimal polynomial is m,

$$\ker N \subseteq \ker N^2 \subseteq \dots \subseteq \ker N^m = \ker N^{m+1} = \dots = V \tag{2.17}$$

Theorem 2.5. The eigenvalue of nilpotent operator is 0.

Proof. Let $|v\rangle$ be an eigenvector, that is $|Nv\rangle = n|v\rangle$. Then $0 = |N^m v\rangle = n^m|v\rangle$. Since $|v\rangle$ is nonzero, $n = 0$. □

2.2 Generalised Eigenvector

Let $A \in End(V)$. Recall the definition of eigenvalue and eigenvector:

$$|(A - \lambda \mathbb{1})v\rangle = 0 \tag{2.18}$$

Therefore, if we define $N_\lambda \coloneqq A - \lambda\mathbb{1}$, eigenspace is just

$$\ker N_\lambda. \tag{2.19}$$

Box 2.3: Generalised Eigevector

Let $A \in \mathrm{End}(V)$. If there is a nonzero vector $|v\rangle$ which satisfies,

$$|(A - \lambda \mathbb{1})^n v\rangle = 0 \qquad \text{where } n \in \mathbb{Z}_+. \tag{2.20}$$

we say $|v\rangle$ is the *generalised eigenvector*.

Then what is n? Since $\ker(A - \lambda\mathbb{1})^n \subseteq \ker(A - \lambda\mathbb{1})^{n+1}$, $\dim(\ker(A - \lambda\mathbb{1})^n) \leq \dim(\ker(A - \lambda\mathbb{1})^{n+1})$. But, we work in finite dimension. Therefore,

$$\exists m \in \mathbb{Z}_+ : \forall n \geq m : \ker\big((A - \lambda\mathbb{1})^m\big) = \ker\big((A - \lambda\mathbb{1})^n\big). \tag{2.21}$$

The m can be found from the minimal polynomial. To show it, we first need the decomposition theorem.

Box 2.4: Decomposition of Vector Space

Let V be a complex vector space and $A \in \mathrm{End}(V)$. Let the minimal polynomial being m. That is,

$$m(A) = (A - \lambda_1)^{m_1} \dots (A - \lambda_r)^{m_r} = 0 \tag{2.22}$$

Then,

$$V = \bigoplus_{i=1}^{r} \ker\big((A - \lambda_i)^{m_i}\big). \tag{2.23}$$

Proof. For notational simplicity, I introduce the notation p_i be a polynomial such that $p_i(A) = (A - \lambda_1)^{m_1}$ and $V_i = \ker p(A)$. Then

$$m = \prod_{i=1}^{r} p_i \tag{2.24}$$

Now, let us define

$$q_i \coloneqq \frac{m}{p_i} = p_1 p_2 \dots p_{i-1} p_{i+1} \dots p_r. \tag{2.25}$$

There are two important properties in this q_i. The first is for $i \neq j$,

$$q_i q_j = \frac{m^2}{p_i p_j} = p_1^2 p_2^2 \dots p_{i-1}^2 p_i p_{i+1}^2 \dots p_{j-1}^2 p_j p_{j+1}^2 = m(\dots) \tag{2.26}$$

In other words, for $i \neq j$, $q_i q_j$ is annihilating polynomial. The second is that collection of them has no common root. That is q_1 has no p_1, q_2 has no p_2. Therefore, $\gcd(q_1, \dots q_n) = 1$. The Bézout identity (for polynomial) tells that there exists polyomials c_i such ehat

$$\sum_i c_i q_i = 1. \tag{2.27}$$

If we put operator in here, we have

$$\sum_i c_i(A) q_i(A) = \mathbb{1}. \tag{2.28}$$

From these properties, we can define the idempotent operator. Let us write $P_i = c_i(A) q_i(A)$. Then it is idemptent:

$$P_i = P_i \mathbb{1} = P_i \sum_j P_j = \sum_j P_i P_j = P_i^2. \tag{2.29}$$

Now recall that $m = p_i q_i$. Therefore,

$$p_i P_i = p_i c_i q_i = c_i m. \tag{2.30}$$

Hence,

$$\forall\, |v\rangle \in V : |p_i(A)(P_i v)\rangle = |(p_i(A) P_i) v\rangle = |c_i(A) m(A) v\rangle = |c_i 0 v\rangle = 0. \tag{2.31}$$

In other words,

$$\operatorname{im} P_i \subseteq \ker p_i = V_i. \tag{2.32}$$

On the other hand,

$$\forall |v\rangle \in V_i : \forall\, j \neq i : |q_j v\rangle = |(\dots p_i \dots) v\rangle = 0. \tag{2.33}$$

Therefore,

$$V_i \subseteq \operatorname{im} P_i, \tag{2.34}$$

with previous relation, yields,

$$V_i = \operatorname{im} P_i. \tag{2.35}$$

Since $\sum_i P_i = \mathbb{1}$,

$$V = \bigoplus_{i=1}^{r} V_i \tag{2.36}$$

$\square$

Remark. The above decomposition make us define generalised eigenspace.

Definition 2.4. Let V be a complex vector space and $A \in \mathrm{End}(V)$. Let the minimal polynomial being m. That is,

$$m(A) = (A - \lambda_1)^{m_1} \dots (A - \lambda_r)^{m_r} = 0 \tag{2.37}$$

A *generalised eigenspace* of an A corresponding to the eigenvalue λ_i is

$$G(\lambda_i) := \ker\big((A - \lambda_1)^{m_1}\big). \tag{2.38}$$

Remark. Now the the decomposition theorem we proved just above can be stated as every complex vector space can be decomposed into a generalised eigenspace.

Now let us find the dimension of generalised eigenspace.

Theorem 2.6. Let V be a complex vector space and $A \in \mathrm{End}(V)$. Each generalised eigenspace reduces A.

Proof. In the above we have shown that $V_i \subseteq \operatorname{im} P_i$. Therefore, V_i is an invariant subspace of P_i. Moreover, since they decompose V, we have

$$V^{\perp} = V_1 \oplus V_2 \oplus \dots \oplus V_{i-1} \oplus V_{i+1} \oplus \dots \oplus V_r. \tag{2.39}$$

From the orthogonality of P_i, we can also see that it is an invariant subspace of P_i. □

Now, we see that in general, the linear endomorphism in complex vector space cannot be diagonalised, but at least we can decompose the vector space with generalised eigenspace. If we try to diagonalise as much as possible, then what we can get? The answer is the Jordan normal form.

Definition 2.5 (Jordan Normal form)**.** The matrix $\mathbf{A} \in \mathrm{Mat}_n(\mathbb{C})$ is called *Jordan normal form* if it is the block diagonal form:

$$\mathbf{A} = \begin{pmatrix} \boxed{J_1(\lambda_1)} & 0 & 0 \\ 0 & \boxed{\ddots} & 0 \\ 0 & 0 & \boxed{J_m(\lambda_r)} \end{pmatrix} \tag{2.40}$$

where each $J_i(\lambda_j) = \lambda_j \mathbb{1} + J_i(0)$ is called *Jordan block* with $J_i(0)$ is the shift of the form:

$$J_i(0) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{2.41}$$

Theorem 2.7 (Jordan Normal Form Theorem)**.** Let V be a complex vectorspace. Every endomorphism $A \in \mathrm{End}(V)$ has a basis which makes the matrix representation of A being a Jordan Normal Form.

Remark. For real vector space, we do not need to have real solutions for the characteristic equation. So, we need to be care on it. There are useful results and theorem in real vector space. But, because of time limit, I skip the real vector space.

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