Endomorphism

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ABSTRACT: References are of course my favorite [1, 2] and additionally, [3–5]. I find [6] is very good.

Contents

1 Endomorphisms

Let us recall the definition of endomorphism.

Definition 1.1 (Endomorphism)**.** A linear map from vector space to itself is called an *endomorphism* or *operator*. We have learnt that the set of endomorphisms of V denoted $\mathrm{End}\,(V)$.

Remark. Recall also that the bijective endomorphism is automorphism.

The importance of endomorphism comes from the fact that its domain and target are the same space. Practically, it means that the matrices representing endomorphisms are square matrices. Moreover, we can work in one specific basis set (in other case, one need two sets of basis for domain and target).

1.1 Matrix Representation of Endomorphisms

For simplicity, we will work in orthonormal basis in the inner product space $(V, \langle\,\cdot\,|\,\cdot\,\rangle)$. The nice property of orthonormal basis is that we can use inner product to make dual basis. That is if $\{|e_i\rangle\}_{i=1}^n$ is the basis for a vector space V, $\{\langle e_i|\}_{i=1}^n$ is the dual basis. Recall the procedure to obtain matrix element of linear map. For $A \in \mathrm{End}(V)$, if we use the Dirac notation, it is

$$\begin{aligned}
&|Ae_i\rangle && A \text{ acts on the basis vector } e_i \text{ in the domain}\\
&\quad = |A_i\rangle && \text{Vector } A_i \text{ is the image.}\\
&\quad = \sum_j {A^j}_i |e_j\rangle && \text{The basis expansion in target space.}
\end{aligned}$$

The matrix element or components of an endomorphism A can be obtained by acting basis and dual basis on it. That is,

$$ {A^j}_i = \langle e_j | A e_i \rangle \tag{1.1}$$

We write the components, but we do not specify the basis yet. Recall that we distinguish two indices i, j to separate the role of domain and target. In Dirac notation, we write the basis (vector of vector space $\mathrm{End}(V)$) with ket and bra mixture to specify it[1].

[1] It is just the tensor product which we will learn later.

Box 1.1: Basis of Endomorphism in Dirac Notation

Let $\{|e_i\rangle\}_{i=1}^n$ be an orthonormal basis for V. Then we choose the corresponding basis for $\mathrm{End}(V)$ as $\{|e_i\rangle\langle e_j|\}_{i,j=1}^n$ such that

$$\forall A \in \mathrm{End}(V) : A = \sum_{i,j} A^j{}_i |e_j\rangle\langle e_i| \qquad \text{where } A^j{}_i = \langle e_j|Ae_i\rangle \tag{1.2}$$

$$=: \sum_{i,j} |e_j\rangle A^j{}_i \langle e_i| \tag{1.3}$$

The action on vector $|v\rangle$ is written

$$|Av\rangle = \sum_{i,j} A^j{}_i |e_j\rangle\langle e_i|v\rangle = \sum_{i,j} A^j{}_i |e_j\rangle v^i. \tag{1.4}$$

Example 1.0.1 (identity)**.** The simplest example is identity operator $\mathbb{1}_V \in \mathrm{Aut}(V) \subset \mathrm{End}(V)$. For identity,

$$\langle e_j|\mathbb{1}_V e_i\rangle = \langle e_j|e_i\rangle = \delta^i{}_j. \tag{1.5}$$

In matrix form, we usually write $\mathbb{1} = \mathrm{diag}[1, 1, \ldots, 1]$. Therefore, the identity matrix is

$$\mathbb{1}_V = \delta^i{}_j |e_i\rangle\langle e_j| = \sum_i |e_i\rangle\langle e_i|. \tag{1.6}$$

Sometimes, while you are solving the problem, you may notice that other basis is easier to solve and you may want to go to the other basis. By the way, what happens to the vector when we change the basis? Nothing. Therefore the basis change is the identity map, which is automorphism.

Box 1.2: Basis Transformation

Let $\mathbb{1} \in \mathrm{Aut}\,(V)$ be the identity operator and $\{|e_i\rangle\}_{i=1}^n$ and $\{|f_i\rangle\}_{i=1}^n$ be two distinct orthonormal basis set for V. Then,

$$|\mathbb{1}e_i\rangle = |e_i\rangle = \sum_j \mathbb{1}^j{}_i |f_j\rangle \tag{1.7}$$

Applying $\langle f_j|$ on both side yields,

$$\langle f_j|e_i\rangle = \mathbb{1}^j{}_i. \tag{1.8}$$

In other words, *change of basis* is just an expression of identity operator in two different basis.

Remark. I use $\mathbb{1}$ symbol to emphasise it is an identity operator. Unfortunately, linear algebra is contaminated by the matrix representation too much, so that $\mathbb{1}$ is usually used for the *identity matrix* which is just $\mathrm{diag}[1, 1, \ldots, 1]$. Therefore, I will use other alphabet to denote it. But please keep in mind that it is just an identity operator.

On the other hand, the change of basis also changes the components. Let $\{|e_i\rangle\}_{i=1}^n$ and $\{|f_i\rangle\}_{i=1}^n$ be two distinct orthonormal basis set for V. Since they are basis, $\forall |v\rangle \in V : \exists! v^1, \dots v^n, v'^1, \dots, v'^n \in \mathbb{K}$ such that

$$\sum_i v^i |e_i\rangle = |v\rangle = \sum_j v'^j |f_j\rangle. \tag{1.9}$$

One can derive how the components changes in many ways. For example, let us use the orthonormality. That is, applying $\langle f_i|$ on both sides (taking inner product) yields

$$\langle f_i| \sum_j v'^j |e_j f\rangle = \sum_j v'^j \langle f_i|f_j\rangle = v'^i = \langle f_i| \sum_j v^j |e_j\rangle = \sum_j v^j \langle f_i|e_j\rangle. \tag{1.10}$$

Therfore,

$$v'^i = \sum_j \langle f_i|e_j\rangle v^j \tag{1.11}$$

Box 1.3: Summary of Change of Basis

Let $\{|e_i\rangle\}_{i=1}^n$ and $\{|f_i\rangle\}_{i=1}^n$ be two distinct orthonormal basis set for V. And let $|v\rangle = \sum v^i |e_i\rangle = \sum v'^j |f_j\rangle$. Then, change of basis

$$|e_i\rangle = \sum_j \langle f_j|e_i\rangle |f_j\rangle \iff |f_j\rangle = \sum_i \langle e_i|f_j\rangle |e_i\rangle \tag{1.12}$$

yields the relation between components:

$$v'^i = \sum_j \langle f_j|e_i\rangle v^j \iff v^j = \sum_i \langle e_i|f_j\rangle v'^i \tag{1.13}$$

The components usually written in matrix form:

$$\mathbf{v}' = \mathbf{S}\mathbf{v} \iff \mathbf{v}' = \mathbf{S}^{-1}\mathbf{v} \tag{1.14}$$

Since the inner product has important role in determining the components, (it defines isomorphism between the basis vector and the dual vector) the change of basis which preserves the inner product is particularly important.

Box 1.4: Orthogonal Transformation

Let us denote the basis transformation O. In the real vector space, the conservation of inner product in different basis can be written

$$\langle Ov|Ov\rangle = \langle v|O^\intercal Ov\rangle = \langle v|v\rangle. \tag{1.15}$$

In other words,

$$O^\intercal O = \mathbb{1}. \tag{1.16}$$

The basis change of this type is called the *orthogonal transformation*.

Remark. In fact, the transformation between orthonormal basis in real vector space is the orthogonal transformation.

Remark. The matrix representation of orthogonal transformation is called the *orthogonal matrix* and it satisfies

$$\delta^j{}_i = \sum_k O^{\intercal j}{}_k O^k{}_i = \sum_k O_k{}^j O^k{}_i \tag{1.17}$$

Box 1.5: Unitary Transformation

Let us denote the basis transformation U. In the complex vector space, the conservation of inner product in different basis can be written

$$\langle Uv|Uv\rangle = \langle v|U^\dagger Uv\rangle = \langle v|v\rangle. \tag{1.18}$$

In other words,

$$U^\dagger U = \mathbb{1}. \tag{1.19}$$

The basis change of this type is called the *unitary transformation*.

Remark. In fact, the transformation between orthonormal basis in complex vector space is the unitary transformation.

Remark. The matrix representation of unitary transformation is called the *unitary matrix* and it satisfies

$$\delta^j{}_i = \sum_k U^{\dagger j}{}_k U^k{}_i = \sum_k \overline{U_k{}^j} U^k{}_i \tag{1.20}$$

Now let us see the change of basis of endomorphisms. There are many ways to derive how the components of endomorphism changes. For example, one can use the identity operator.

$$A'^j{}_i = \langle f_j|Af_i\rangle = \langle f_j|\mathbb{1}A\mathbb{1}f_i\rangle = \sum_{k,l}\langle f_j|e_k\rangle\langle e_k|Ae_l\rangle\langle e_l|f_i\rangle = \sum_{k,l}\langle f_j|e_k\rangle A^k{}_l\langle e_l|f_i\rangle \tag{1.21}$$

Box 1.6: Similarity Transformation

The components of linear map changes

$$A'^j{}_i = \sum_{k,l}\langle f_j|e_k\rangle A^k{}_l\langle e_l|f_i\rangle \tag{1.22}$$

Using the transformation matrix $\mathbf{S} = (\langle f_j|e_k\rangle)$, it can be written

$$\mathbf{A}' = \mathbf{SAS^{-1}} \iff \mathbf{A} = \mathbf{S^{-1}A'S} \tag{1.23}$$

This transformation is called the *similarity transformation*.

Remark. If the basis change is orthogonal transformation $\mathbf{S}^{-1} = \mathbf{S}^\intercal$. If the basis change is unitary transformation, $\mathbf{S}^{-1} = \mathbf{S}^\dagger$.

2 Basis Independent things

Recall that basis change is just identity map. Therefore, there should be some quantities which are invariant under the basis change. That invariant quantities are the true important thing in the linear algebra (and physics). In this section, we will see the operation in the matrix which is invariant under the basis change.

2.1 Trace

One famous operation on the square matrix you might also be familiar with is the trace.

Box 2.1: Trace of a Matrix

A *trace* of a square matrix $\mathbf{A} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is the sum of its diagonal elements:

$$\mathrm{tr}A := \sum_i A^i{}_i. \tag{2.1}$$

Theorem 2.1. It has the following properties:
- ⋄ $\forall \mathbf{A} \in \mathrm{Mat}_{n\times n} : \mathrm{tr}\mathbf{A}^\intercal = \mathrm{tr}\mathbf{A}$;
- ⋄ $\forall\, a \in \mathbb{K} : \forall \mathbf{A}, \mathbf{B} \in \mathrm{Mat}_{n\times n} : \mathrm{tr}(a\mathbf{A} + \mathbf{B}) = a\mathrm{tr}\mathbf{A} + \mathrm{tr}\mathbf{B}$
- ⋄ $\forall \mathbf{A}, \mathbf{B} \in \mathrm{Mat}_{n\times n} : \mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$.

Proof. The first and second is trivial. The third is

$$\mathrm{tr}(\mathbf{AB}) = \sum_i (\mathbf{AB})^i{}_i = \sum_i \Big(\sum_j A^i{}_j B^j{}_i\Big) = \sum_j \Big(\sum_i B^j{}_i A^i{}_j\Big) = \sum_j (\mathbf{BA})^j{}_j = \mathrm{tr}(\mathbf{BA}).$$

□

Lemma 2.2. From the second theorem, we can derive the following lemma:
- ⋄ *cyclic:* $\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{BCA}) = \mathrm{tr}(\mathbf{CAB})$.;
- ⋄ *invarance under similarity transform:* $\mathrm{tr}(\mathbf{SAS^{-1}}) = \mathrm{tr}(\mathbf{A})$.

Recall that the similarity transformation is the matrix representation of component transformation by basis change. Therefore, if we calculate trace of an endomorphism in some basis, it is the same for every other choice of basis. Moreover, it is linear map.

Box 2.2: Trace of an Endomorphism

Let $A \in \mathrm{End}(V)$. The *trace* of endomorphism A is a linear map which is defined to be the trace of matrix representation of it in *any* basis.

$$\mathrm{tr} : \mathrm{End}(V) \to \mathbb{K} \tag{2.2}$$

$$A \mapsto \mathrm{tr}A := \mathrm{tr}\mathbf{A} = \sum_i A^i{}_i. \tag{2.3}$$

where

$$A = \sum_{i,j} A^i{}_j |e_i\rangle\langle e_j|. \tag{2.4}$$

for some basis set $\{e_i\}_{i=1}^n$ for V.

Definition 2.1 (Inner product of endomorphisms on real vector space)**.** A map

$$g : \mathrm{End}(V) \times \mathrm{End}(V) \to \mathbb{R} \tag{2.5}$$
$$(A, B) \mapsto g(A, B) \coloneqq \mathrm{tr}(A^\intercal B) \tag{2.6}$$

is an inner product (symmetric bilinear map).

Definition 2.2 (Inner product of endomorphisms on complex vector space)**.** A map

$$g : \mathrm{End}(V) \times \mathrm{End}(V) \to \mathbb{C} \tag{2.7}$$
$$(A, B) \mapsto g(A, B) \coloneqq \mathrm{tr}(A^\dagger B) \tag{2.8}$$

is an inner product (sesquilinear bilinear map).

2.2 Determinant

Another famous operation on the square matrix is the determinant.

Box 2.3: Determinant of a Matrix

A *determinant* of a square matrix $\mathbf{A} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is

$$\det \mathbf{A} = \sum_{i_1,\dots,i_n}^{n} \varepsilon_{i_1,\dots,i_n} {A^{i_1}}_1 \dots {A^{i_n}}_n. \tag{2.9}$$

Definition 2.3 (submatrix)**.** A *submatrix* of a matrix $\mathbf{A} \in \mathrm{Mat}_{m\times n}(\mathbb{K})$ is a matrix which is defined by eliminating some columns and rows.

Remark. It can be defined in every matrix not only square matrix.

Example 2.2.1.

$$\begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}. \tag{2.10}$$

Definition 2.4 (Minor)**.** A *minor* of a square matrix $\mathbf{A} \in \mathrm{Mat}_n(\mathbb{K})$ is a determinant of the submatrix of $\mathbf{A}$. A *first minor* or (i, j)*-minor*, denoted ${M^i}_j$, is the determinant of the submatrix of $\mathbf{A}$ which is defined by ith row and jth column.

Definition 2.5 (cofactor)**.** A *cofactor* of (i, j) component of square matrix is

$$(-1)^{i+j} {M^i}_j \tag{2.11}$$

A *cofactor matrix* is the matrix whose components are cofactors. That is

$${C^i}_j = (-1)^{i+j} {M^i}_j \tag{2.12}$$

Theorem 2.3 (Laplace Expansion)**.**

$$\begin{aligned} \det \mathbf{A} &= \sum_i (-1)^{i+j} {M^i}_j {A^i}_j \qquad (\forall j \in \{1, \dots, n\}) \\ &= \sum_i (-1)^{i+j} {M^i}_j {A^i}_j \qquad (\forall i \in \{1, \dots, n\}). \end{aligned}$$

Theorem 2.4. Let $\det \mathbf{A} \in \mathrm{Mat}_n(\mathbb{K})$ the determinant has the following properties:

- ◇ $\det \mathbf{A}^\intercal = \det \mathbf{A}$;
- ◇ $\det \mathbf{A}^\dagger = \overline{\det \mathbf{A}} = \det \overline{A}$;

Proof. It can be done by transposition of Levi-Civita symbol and matrix indices upside down:

$$\det \mathbf{A} = \sum_{i_1,\dots,i_n}^{n} \varepsilon_{i_1,\dots,i_n} A^{i_1}{}_1 \dots A^{i_n}{}_n = \sum_{i_1,\dots,i_n}^{n} \varepsilon^{i_1,\dots,i_n} A^1{}_{i_1} \dots A^n{}_{i_n} \tag{2.13}$$

□

Definition 2.6. A square matrix $\mathbf{A} \in \mathrm{Mat}_n(\mathbb{K})$ is called a *upper triangular matrix* if

$$\forall i > j \in \{1, \dots, n\} : \qquad A^i{}_j = 0; \tag{2.14}$$

and called the *lower triangular matrix* if

$$\forall i < j \in \{1, \dots, n\} : \qquad A^i{}_j = 0. \tag{2.15}$$

Both cases are simply called the *trangular matrix*.

Theorem 2.5. For triangular matrix $\mathbf{A} \in \mathrm{Mat}_n(\mathbb{K})$,

$$\det \mathbf{A} = \prod_i A^i{}_i. \tag{2.16}$$

Proof. For any exchange, $A^i{}_j = 0$. □

Lemma 2.6. For the diagonal matrix $\mathbf{D} = \mathrm{diag}[\lambda_1, \dots, \lambda_n]$,

$$\det \mathbf{D} = \prod_i \lambda_i. \tag{2.17}$$

Theorem 2.7 (Leibniz formula for determinants)**.** For square matrix $\mathbf{A} \in \mathrm{Mat}_n(\mathbb{K})$,

$$\det \mathbf{A} \varepsilon_{j_1,\dots,j_n} = \sum_{i_1,\dots,i_n} \varepsilon_{i_1,\dots,i_n} A^{i_1}{}_{j_1} \dots A^{i_n}{}_{j_n}. \tag{2.18}$$

Proof. Recall that $\varepsilon_{i_1,\dots,i_n} = \mathrm{sgn}\left(P^{1,\dots,n}_{i_1,\dots,i_n}\right)$.

$$\begin{aligned}
\det \mathbf{A} &= \varepsilon_{i_1,\dots,i_n} A^{i_1}{}_1 \dots A^{i_n}{}_n \\
&= \varepsilon_{i_1,\dots,i_n} \mathrm{sgn}\left(P^{1,\dots,n}_{j_1,\dots,j_n}\right) A^{i_1}{}_{j_1} \dots A^{i_n}{}_{j_n} \\
&= \varepsilon_{i_1,\dots,i_n} \varepsilon_{j_1,\dots,j_n} A^{i_1}{}_{j_1} \dots A^{i_n}{}_{j_n}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\det \mathbf{A} \varepsilon_{j_1,\dots,j_n} &= \varepsilon_{i_1,\dots,i_n} (\varepsilon_{j_1,\dots,j_n})^2 A^{i_1}{}_{j_1} \dots A^{i_n}{}_{j_n} \\
&= \varepsilon_{i_1,\dots,i_n} A^{i_1}{}_{j_1} \dots A^{i_n}{}_{j_n}
\end{aligned}$$

□

Theorem 2.8. Let $\mathbf{A}, \mathbf{B} \in \mathrm{Mat}_n(\mathbb{K})$. Then

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}). \tag{2.19}$$

Proof.

$$\begin{aligned}\det(\mathbf{AB}) &= \varepsilon_{i_1,\ldots,i_n} (AB)^{i_1}{}_1 \ldots (AB)^{i_n}{}_n \\ &= \varepsilon_{i_1,\ldots,i_n} A^{i_1}{}_{j_1} B^{j_1}{}_1 \ldots A^{i_n}{}_{j_n} B^{j_n}{}_n \\ &= \det \mathbf{A} \varepsilon_{j_1,\ldots,j_n} B^{j_1}{}_1 \ldots B^{j_n}{}_n \\ &= (\det \mathbf{A})(\det \mathbf{B}).\end{aligned}$$

In the third equality, we used the Leibniz formula for determinants. □

Theorem 2.9. If there is an inverse of the square matrix,

$$\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}. \tag{2.20}$$

Proof.

$$1 = \det \mathbb{1} = \det(\mathbf{A}\mathbf{A}^{-1}) = (\det \mathbf{A})(\det \mathbf{A}^{-1}). \tag{2.21}$$

□

Definition 2.7 (Singular Matrix)**.** A square matrix $\mathbf{A} \in \mathrm{Mat}_n(\mathbb{K})$ is called *singular* matrix if $\det \mathbf{A} = 0$. If it is not, it is called *regular* or *invertible* or .

Remark. The determinant is used to check the existence of inverse. In fact, the inverse of matrix can be written in the determinant.

Definition 2.8 (adjugate matrix)**.** An *adjugate matrix* is the transpose of cofactor matrix.

$$\hat{A}^i{}_j = C^j{}_i \tag{2.22}$$

Theorem 2.10. Let $\mathbf{A} \in \mathrm{Mat}_n(\mathbb{K})$ be an invertible matrix. Then,

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \hat{\mathbf{A}}. \tag{2.23}$$

Definition 2.9 (block matrix)**.** A block matrix is the matrix which can be partitioned by the smaller matrices for chosen row and column. See following example.

Example 2.10.1. Let $\mathbf{M} \in \mathrm{Mat}_{m\times n}(\mathbb{K})$. And let $\mathbf{A} \in \mathrm{Mat}_{m_1\times n_1}(\mathbb{K}), \mathbf{B} \in \mathrm{Mat}_{m_1\times n_2}(\mathbb{K})\mathbf{C} \in \mathrm{Mat}_{m_2\times n_1}(\mathbb{K})\mathbf{D} \in \mathrm{Mat}_{m_2\times n_2}(\mathbb{K})$ with $m_1 + m_2 = m$, $n_1 + n_2 = n$. If

$$\begin{aligned} &\forall i \in \{1, \ldots, m_1\}, j \in \{1, \ldots, n_1\}: && M^i{}_j = A^i{}_j \\ &\forall i \in \{1, \ldots, m_1\}, j \in \{1, \ldots, n_2\}: && M^i{}_{j+n_1} = B^i{}_j \\ &\forall i \in \{1, \ldots, m_2\}, j \in \{1, \ldots, n_1\}: && M^{i+m_1}{}_j = C^i{}_j \\ &\forall i \in \{1, \ldots, m_2\}, j \in \{1, \ldots, n_2\}: && M^{i+m_1}{}_{j+n_1} = D^i{}_j. \end{aligned}$$

We call $\mathbf{M}$ is a block matrix and denotes

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \tag{2.24}$$

Lemma 2.11. For the square block matrix of the form

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \tag{2.25}$$

where $\mathbf{0}$ is the zero matrix and $\mathbf{A}, \mathbf{D}$ are square matrix,

$$\det \mathbf{M} = (\det \mathbf{A})(\det \mathbf{D}). \tag{2.26}$$

Proof. If $\mathbf{A}$ is singular, $\det \mathbf{A} = 0$ and it results $\det \mathbf{M} = 0$. If $\mathbb{A}$ is invertible,

$$\begin{aligned} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} &= \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbb{1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbb{1} \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \det \mathbf{M} &= \det \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbb{1} \end{pmatrix} \det \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \det \begin{pmatrix} \mathbb{1} & \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbb{1} \end{pmatrix} \\ &= (\det \mathbf{A})(\det \mathbf{D}) \end{aligned}$$

Since $\det \mathbb{1} = 1$. □

Theorem 2.12. The determinant of a square matrix is invariant under the similiarity transform.

Proof. Let $\mathbf{S}$ be a transformation matrix. Then

$$\det(\mathbf{SAS}^{-1}) = (\det \mathbf{S})(\det \mathbf{A})(\det \mathbf{S}^{-1}) = \det \mathbf{A}. \tag{2.27}$$

□

In other words, if we calculate determinant in one basis, whatever we choose other basis, the determinant is always the same.

Box 2.4: Determinant of an Endomorphism

Let $A \in \mathrm{End}(V)$. The *determinant* of endomorphism A is a linear map which is defined to be the trace of matrix representation of it in *any* basis.

$$\det\ : \mathrm{End}(V) \to \mathbb{K} \tag{2.28}$$

$$A \mapsto \det A \coloneqq \det \mathbf{A} = \sum_{i_1,\ldots,i_n}^{n} \varepsilon_{i_1,\ldots,i_n} A^{i_1}{}_1 \ldots A^{i_n}{}_n. \tag{2.29}$$

where

$$A = \sum_{i,j} A^i{}_j |e_i\rangle\langle e_j|. \tag{2.30}$$

for some basis set $\{e_i\}_{i=1}^n$ for V.

Remark. There are some properties I do not prove or not introduced here. In fact, determinant can be also understood as a multiliner form. I keep it for later.

2.2.1 Determinant and Trace

The relation between $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$ and $\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}\mathbf{A} + \mathrm{tr}\mathbf{B}$ reminiscents the relation exp or log in the sense that multiplication and addition are related. In this section, let's explore the relation between the determinant and trace.

Theorem 2.13. Let $A \in \mathrm{End}(V)$, and $0 < \epsilon \ll 1$ be a small number.

$$\det(\mathbb{1} + \epsilon A) = 1 + \epsilon\,\mathrm{tr}A \tag{2.31}$$

Proof. Let us assume we work in some basis. Then,

$$\begin{aligned}
\det(\mathbb{1} + \epsilon A) &= \sum_{i_1,\ldots,i_n}^{n} \varepsilon_{i_1,\ldots,i_n} (\delta^{i_1}{}_1 + \epsilon A^{i_1}{}_1) \ldots (\delta^{i_n}{}_n + \epsilon A^{i_n}{}_n) \\
&= \sum_{i_1,\ldots,i_n}^{n} \varepsilon_{i_1,\ldots,i_n} \delta^{i_1}{}_1 \ldots \delta^{i_n}{}_n \\
&\quad + \epsilon \sum_{k=1}^{n} \sum_{i_1,\ldots,i_n}^{n} \varepsilon_{i_1,\ldots,i_n} \delta^{i_1}{}_1 \ldots \delta^{i_{k-1}}{}_{k-1} A^{i_k}{}_k \delta^{i_{k+1}}{}_{k+1} \ldots \delta^{i_n}{}_n + \mathcal{O}(\epsilon^2) \\
&= \det \mathbb{1} + \epsilon \sum_{k=1}^{n} \sum_{i_k=1}^{n} \varepsilon_{1,\ldots,k-1,i_k,k,\ldots,n} A^{i_k}{}_k + \mathcal{O}(\epsilon^2) \\
&= \det \mathbb{1} + \epsilon \sum_{k=1}^{n} \varepsilon_{1,\ldots,n} A^{k}{}_k + \mathcal{O}(\epsilon^2) \\
&= \det \mathbb{1} + \epsilon\,\mathrm{tr}\mathbf{A}.
\end{aligned}$$

□

Lemma 2.14. Let $\mathbf{A}(t) \in \mathrm{Mat}_{n\times n}$ be a matrix whose elements $A^i{}_j(t)$ are differentiable function of t. Moreover, if $\mathbf{A}(0) = \mathbb{1}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\det \mathbf{A}\big)|_{t=0} = \lim_{\epsilon\to 0} \frac{\mathbf{A}(\epsilon) - \mathbf{A}(0)}{\epsilon} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(0). \tag{2.32}$$

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References

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